To Camilla

What good is it for a man
to gain the whole world,
yet forfeit his life?
Mc. 8, 36

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Andrea Pascucci

PDE and Martingale Methods in Option Pricing

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Preface

This book gives an introduction to the mathematical, probabilistic and numerical methods used in the modern theory of option pricing. It is intended as a textbook for graduate and advanced undergraduate students, but I hope it will be useful also for researchers and professionals in the financial industry.

Stochastic calculus and its applications to the arbitrage pricing of financial derivatives form the main theme. In presenting these, by now classic, topics, the emphasis is put on the more quantitative rather than economic aspects. Being aware that the literature in this field is huge, I mention the following incomplete list of monographs whose contents overlap with those of this text: in alphabetic order, Avellaneda and Laurence [14], Benth [43], Björk [47], Dana and Jeanblanc [84], Dewynne, Howison and Wilmott [340], Dothan [100], Duffie [102], Elliott and Kopp [120], Epps [121], Follmer and Schied [134], Glasserman [158], Huang and Litzenberger [171], Ingersoll [178], Karatzas [200; 202], Lamberton and Lapeyre [226], Lipton [239], Merton [252], Musiela and Rutkowski [261], Neftci [264], Shreve [310; 311], Steele [315], Zhu, Wu and Chern [349].

What distinguishes this book from others is the attempt to present the matter by giving equal weight to the probabilistic point of view, based on the martingale theory, and the analytical one, based on partial differential equations. The present book does not claim to describe the latest developments in mathematical finance: that target would indeed be very ambitious, given the speed of progress of research in the field. Instead, I have chosen to develop some of the essential ideas of the classical pricing theory to devote space to the fundamental mathematical and numerical tools when they arise. Thus I hope to provide a sound background of basic knowledge which may facilitate the independent study of newer problems and more advanced models.

The theory of stochastic calculus, for continuous and discontinuous processes, constitutes the bulk of the book: Chapters 3 on stochastic processes, 4 on Brownian integration and 9 on stochastic differential equations may form the material for an introductory course on stochastic calculus. In these chapters, I have constantly sought to combine the theoretical concepts to the insight on the financial meaning, in order to make the presentation less abstract and more motivated: in fact many theoretical concepts naturally lend themselves to an intuitive and meaningful economic interpretation.

The origin of this book can be traced to courses on option pricing which I taught at the master program in Quantitative Finance of the University of Bologna, which I have directed with Sergio Polidoro since its beginning, in 2004. I wrote the first version as lecture notes for my courses. During these years, I substantially improved and extended the text with the inclusion of sections on numerical methods and the addition of completely new chapters on stochastic calculus for jump processes and Fourier methods. Nevertheless, during these years the original structure of the book remained essentially unchanged.

I am grateful to many people for the suggestions and helpful comments with which supported and encouraged the writing of the book: in particular I would like to thank several colleagues and PhD students for many valuable suggestions on the manuscript, including David Applebaum, Francesco Caravenna, Alessandra Cretarola, Marco Di Francesco, Piero Foscari, Paolo Foschi, Ermanno Lanconelli, Antonio Mura, Cornelis Oosterlee, Sergio Polidoro, Valentina Prezioso, Enrico Priola, Wolfgang Runggaldier, Tiziano Vargiolu, Valeria Volpe. I also express my thanks to Rossella Agliardi, co-author of Chapter 13, and to Matteo Camaggi for helping me in the translation of the book.

It is greatly appreciated if readers could forward any errors, misprints or suggested improvements to: `andrea.pascucci@unibo.it`
Corrections received after publication will be posted on the website:
`http://www.dm.unibo.it/~pascucci/`

Bologna, November 2010 *Andrea Pascucci*

Contents

General notations

- $\mathbb{N} = \{1, 2, 3, \dots\}$ is the set of natural numbers
- $\mathbb{N}_0 = \{0, 1, 2, 3, \dots\}$ is the set of non-negative integers
- $\mathbb{Q}$ is the set of rational numbers
- $\mathbb{R}$ is the set of real numbers
- $\mathbb{R}_{>0} =]0, +\infty[$
- $\mathbb{R}_{\geq 0} = [0, +\infty[$
- $\mathcal{S}_T =]0, T[\times \mathbb{R}^N$ is a strip in $\mathbb{R}^{N+1}$
- $\mathscr{B} = \mathscr{B}(\mathbb{R}^N)$ is the Borel σ-algebra in $\mathbb{R}^N$
- $|H|$ or $m(H)$ denote the Lebesgue measure of $H \in \mathscr{B}$
- $\mathbb{1}_H$ is the indicator function of H, p. 606
- $\partial_x = \frac{\partial}{\partial x}$ is the partial derivative with respect to x

For any $a, b \in \mathbb{R}$,

- $a \wedge b = \min\{a, b\}$
- $a \vee b = \max\{a, b\}$
- $a^+ = \max\{a, 0\}$
- $a^- = \max\{-a, 0\}$

For any $N \times d$-matrix $A = (a_{ij})$,

- A^* is the transpose of A
- $\mathrm{tr}A$ is the trace of A
- $\mathrm{rank}A$ is the rank of A
- $|A| = \sqrt{\sum_{i=1}^{N} \sum_{j=1}^{d} a_{ij}^2}$
- $\|A\| = \sup_{|x|=1} |Ax|$

Note that $\|A\| \leq |A|$. The point $x \in \mathbb{R}^N$ is identified with a column vector $N \times 1$ and

$$x^* y = \langle x, y \rangle = x \cdot y = \sum_{i=1}^{N} x_i y_i$$

denotes the Euclidean scalar product in $\mathbb{R}^N$.

Depending on the context, $\mathcal{F}$ denotes the Fourier transform or the σ-algebra of a probability space. The Fourier transform of a function f is denoted by $\hat{f}$.

Shortenings

- $A := B$ means that "by definition, A equals B"
- r.v. = random variable
- s.p. = stochastic process
- a.s. = almost surely
- a.e. = almost everywhere
- i.i.d. = independent and identically distributed (referred to random variables)
- mg = martingale
- PDE = Partial Differential Equation
- SDE = Stochastic Differential Equation

Function spaces

- $m\mathscr{B}$: space of $\mathscr{B}$-measurable functions, p. 608
- $m\mathscr{B}_b$: space of bounded functions in $m\mathscr{B}$, p. 608
- BV: space of functions with bounded variation, p. 127
- Lip: space of Lipschitz continuous functions, p. 679
- $\mathrm{Lip}_{\mathrm{loc}}$: space of locally Lipschitz continuous functions, p. 679
- C^k: space of functions with continuous derivatives up to order $k \in \mathbb{N}_0$
- C_b^k: space of functions in C^k bounded together with their derivatives
- $C^{k+\alpha}$: space of functions differentiable up to order $k \in \mathbb{N}_0$ with partial derivatives that are Hölder continuous of exponent $\alpha \in]0,1[$
- $C_{\mathrm{loc}}^{k+\alpha}$: space of functions differentiable up to order $k \in \mathbb{N}_0$ with partial derivatives that are locally Hölder continuous of exponent $\alpha \in]0,1[$
- C_0^∞: space of test functions, i.e. smooth functions with compact support, p. 678
- $C^{1,2}$: space of functions $u = u(t,x)$ with continuous second order derivatives in the "spatial" variable $x \in \mathbb{R}^N$ and continuous first order derivative in the "time" variable t, p. 631
- C_P^α: space of parabolic Hölder continuous functions of exponent α, p. 258
- L^p: space of functions integrable of order p
- L_{loc}^p: space of functions locally integrable of order p
- $W^{k,p}$: Sobolev space of functions with weak derivatives up to order k in L^p, p. 679
- S^p: parabolic Sobolev space of functions with weak second order derivatives in L^p, p. 265

Spaces of processes

- $\mathbb{L}^p$: space of progressively measurable processes in $L^p([0,T] \times \Omega)$, p. 141
- $\mathbb{L}^p_{\text{loc}}$: space of progressively measurable processes X such that $X(\omega) \in L^p_{\text{loc}}([0,T])$ for almost any ω, p. 159
- $\mathcal{A}_{\text{c}}$: space of continuous processes $(X_t)_{t\in[0,T]}$, $\mathcal{F}_t$-adapted and such that

$$[\![X]\!]_T = \sqrt{E\left[\sup_{0\le t\le T} X_t^2\right]}$$

  is finite, p. 280
- $\mathscr{M}^2$: linear space of right continuous martingales $(M_t)_{t\in[0,T]}$ such that $M_0 = 0$ a.s. and $E\left[M_T^2\right]$ is finite, p. 115
- $\mathscr{M}^2_{\text{c}}$: linear subspace of the continuous martingales of $\mathscr{M}^2$, p. 115
- $\mathscr{M}_{\text{c,loc}}$: space of continuous local martingales M such that $M_0 = 0$ a.s., p. 161