The Mathematical Legacy of Srinivasa Ramanujan

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*On a height he stood that looked towards
greater heights.
Our early approaches to the Infinite
Are sunrise splendours on a marvellous verge
While lingers yet unseen the glorious sun.
What now we see is a shadow of what must
come.*

Sri Aurobindo, Savitri 1.4

*How I wish I could show you the world
through my eyes.*

Vivekananda

Preface

22 December 2012 marks the 125th birth anniversary of the Indian mathematician Srinivasa Ramanujan. Being largely self-taught, he emerged from extreme poverty to become one of 20th century's most influential mathematicians. His story is a phenomenal "rags to mathematical riches" story. In his short life, he had a wealth of ideas that have transformed and reshaped 20th century mathematics. These ideas continue to shape mathematics of the 21st century.

This book is meant to be a panoramic view of his essential mathematical contributions. It is not an encyclopedic account of Ramanujan's work. Rather, it is an informal account of some of the major developments that emanated from his work in the 20th and 21st centuries. The twelve essays focus on a subset of his significant papers and show how these papers shaped the course of modern mathematics.

These essays are based on lectures given by the authors over the years at the Chennai Mathematical Institute, Harish-Chandra Research Institute, IISER (Kolkata), IISER (Bhopal), IIT (Powai), IIT (Chennai), Institute for Mathematical Sciences (Chennai), and the Tata Institute for Fundamental Research (Mumbai) as well as Queen's University, the Fields Institute, and the University of Toronto. The lectures were given so that the material is accessible to undergraduates and graduate students. We have striven to not be too technical. At the same time, we tried to convey some depth of the mathematical theories emerging from the work of Ramanujan. Surely, it is impossible to be comprehensive in such a mammoth task. Still, we hope that the reader will see how the vast landscape of Ramanujan's garden has blossomed over the past century.

Toronto, Canada

M. Ram Murty
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