Treks into Intuitive Geometry

Jin Akiyama • Kiyoko Matsunaga

Treks into Intuitive Geometry

The World of Polygons and Polyhedra

Jin Akiyama
Tokyo University of Science
Tokyo, Japan

Kiyoko Matsunaga
Yokohama, Kanagawa, Japan

ISBN 978-4-431-55841-5 ISBN 978-4-431-55843-9 (eBook)
DOI 10.1007/978-4-431-55843-9

Library of Congress Control Number: 2015956020

Springer Tokyo Heidelberg New York Dordrecht London
© Springer Japan 2015

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made.

Printed on acid-free paper

Springer Japan KK is part of Springer Science+Business Media (www.springer.com)

Preface

Mathematical truth remains uninfluenced by speculation or perspective, and its truth and reality transcend time and space. Mathematical theories do not allow for vagueness or compromise. Therefore, once a theory has been constructed on the basis of its proof, it cannot be altered by anyone. Furthermore, everyone can admire the beautiful fruit borne of mathematical theories, no matter what their race, age, religious beliefs, historical views or perception of life may be.

The authors of this book aim to demonstrate the elegant mathematical proofs of various theorems to as many people as possible, especially to university students. Additionally, this book was written so that readers can enjoy discovering new theories, regardless of their simplicity.

One of the motivating factors behind this book was the fact that we have been involved in mathematical programs on television and radio for over two decades and have found that many viewers and listeners were tired of the dull math taught in school. We strove to engage the audience's senses to demonstrate how wonderful math can be. In the programs we tried as much as possible to avoid any top-down teaching methods. We wanted viewers to discover the processes by which the founders of the theories came to their various conclusions – their trials, errors, and tribulations.

We have always kept in mind the words of Ernest Rutherford, “If you can’t explain a result in simple, nontechnical terms, then you don’t really understand it.” In other words, “you don’t really understand it” means not that your result is wrong, but that you do not fully understand its origin, meaning, or implications.

Fortunately, as a result of our approach, many viewers have told us that for the first time in their lives they understood the joy of mathematics. However, there are still many theories that are difficult to grasp no matter how much they are broken down. Generally speaking, in order to understand complex theories it is imperative to have a high level of knowledge and understanding of abstract concepts. At the same time we cannot enjoy or be grateful for such theories unless we fully recognize the significance of the theories themselves.

There are large, beautiful, unusual flowers on the peaks of high mountains, but there are also common flowers such as violets, bellflowers and dandelions that are just as beautiful on the small hills near our houses. Similarly, in this book we highlight and focus on wisdom taken from daily life – such as examples from various works of art, traditional crafts, patterns that appear in nature, music, and mathematical mechanisms in techniques that craftspeople use. This book is written in a style that unearths the mathematical theories buried in our everyday lives. Our goal is for our readers to enjoy the process of applying mathematical rules to beautiful art and design by highlighting examples of wonders and mysteries from our daily lives. To fulfill these aims, this book deals with polygons and polyhedra that can be found

around us. There are detailed explanations concerning their nets, cross sections, surface areas, and volumes – as well as their filling properties, their transformations, and their decomposabilities.

In this book, Kyuta – a student – is led by a geometry researcher –Gen – through a forest of geometry. Through a series of discussions they solve mathematical problems step by step. They trek through this vivid forest to dig up mysterious treasure boxes.

Intuitive geometry is not a field well explored within mathematics. The term “Intuitive Geometry” does not appear in the AMS subject classification. It was coined by Hungarian mathematician László Fejes Tóth to refer to the kind of geometry which, in Hilbert’s words, can be explained to and can appeal to the “man on the street.”

This book allows people to enjoy intuitive geometry casually and instinctively. It does not require more than high school level knowledge, but does call for a sense of wonder, intuition, and mathematical maturity.

Now, let us begin our trek to hunt for mathematical treasures in this forest of geometry, where beautiful flowers bloom and small woodland creatures await!

Acknowledgement

The study of intuitive geometry and discrete geometry in Japan has a history dating back to the 1980s.

In those days, David Avis, Vašek Chvátal, Michel Deza, Paul Erdős, Peter Frankl, Ron Graham and Jorge Urrutia played the role of missionaries by giving lectures on this academic field at their respective universities.

Thanks to their efforts, the first JCDCG (Japan Conference on Discrete and Computational Geometry) was held in Tokyo in 1997, and since then it has been held annually; at the time of this writing up to the 19th conference this year (2015).

Guest speakers at these conferences included Takao Asano, Tetsuo Asano, David Avis, Imre Barany, Sergey Bereg, William Y. C. Chen, Vašek Chvátal, William Cook, Erik Demaine, Martin Demaine, Nikolai Dolbilin, Rudolf Fleischer, Greg Frederickson, Ferran Hurtado, Hiroshi Imai, Hiro Ito, Mikio Kano, Naoki Katoh, Ken-ichi Kawarabayashi, David Kirkpatrick, Stefan Langerman, Xueliang Li, Hiroshi Maehara, Jiri Matoušek, Alberto Márquez, Frank Nielsen, Joseph O'Rourke, János Pach, Rom Pinchasi, Pedro Ramos, Eduardo Rivera-Campo, Akira Saito, Jonathan Shewchuk, Gyula Solymosi, William Steiger, Kokichi Sugihara, Endre Szemerédi, Xuehou Tan, Takeshi Tokuyama, Godfried Toussaint, Géza Tóth, Ryuhei Uehara, Jorge Urrutia, and Chuanming Zong as well as many other respected speakers. We owe many topics dealt with in this book to the series of conferences.

Additionally, research meetings specialized in intuitive geometry were started by Jin-ichi Itoh in 2009 at Kumamoto University. At the time of this writing, they are holding their seventh meeting this year (2015). With international support from various people, this field has also been able to grow in Japan and many universities have now courses on this subject.

This book was actually written based on the lecture notes of the Intuitive Geometry and Discrete Geometry course taught at Tokyo University of Science.

The course was programmed for junior and senior university students majoring in math. In the class, many of the models and works that appeared in this book were used as visual aids, in order to enhance the understanding of abstract concepts. Mathematical exercises enabled the students to utilize the theories in this book to create mathematical works. It was necessary for students to go over the theories they learned repeatedly in order to complete the artwork that required mathematical accuracy.

Although it had a somewhat different atmosphere than that of a normal math class, many of the students were absorbed in their work. I would like to express my thanks to my dedicated students who completed artworks of mathematical beauty.

Last but not least, we would like to extend thanks for all the support we received in completing this book. We would like to express our deep and sincere gratitude to Vašek Chvátal,

Alex Cole, Erik Demaine, Martin Demaine, Agnes Garciano, Mark Goldsmith, Mikio Kano, Keiko Kotani, Stefan Langerman, Hiroshi Maehara, Pauline Ann Mangulabnan, Gisaku Nakamura, Chie Nara, Amy Ota, David Rappaport, Mari-Jo Ruiz, Ikuro Sato, Jorge Urrutia, Margaret Schroeder and Teruhisa Sugimoto who corrected the manuscript and gave valuable comments.

We would also like to gratefully acknowledge Kaoru Yumi, who provided many illustrations and pictures of art; Yasuyuki Yamaguchi and Hynwoo Seong, who created the complicated drawings and diagrams; and Toru Takemura, Etsuko Watanabe, and Hidetoshi Okazaki, who contributed an enormous amount of illustrations and typing.

Contents

Chapter 1	Art From Tiling Patterns	1
1.	Geometric Patterns	1
2.	Tilings	9
3.	Conway Criterion	17
4.	Math Behind Masterpieces	24
	Appendix 1.1.1	30
	Appendix 1.2.1	33
	References	34
Chapter 2	The Tile-Maker Theorem and Its Applications to Art and Designs	35
1.	How to Draw Repeated Patterns Like Escher's	35
2.	Tile-Makers	38
3.	Characteristics of a Net of an Isotetrahedron	41
4.	Polyhedral Stampers	45
5.	Dihedral Stampers and Dihedral Tile-Maker Theorem	49
6.	From Shark Heads to Tile-Makers	53
7.	Exhibition of Collage Art	54
	Appendix 2.3.1	60
	Appendix 2.4.1	62
	Appendix 2.5.1	63
	References	64
Chapter 3	Patchwork	67
1.	Patchwork — Equidecomposable Pairs of Polygons	67
2.	How to Turn a Polygon into a Rectangle	74
3.	Theorem of Equidecomposability	78
	References	80
Chapter 4	Reversible Pairs of Figures	81
1.	Envelope Magic	81
2.	The Envelope Magic Trick	84
3.	Trapezoidal Envelope Magic	88
4.	Dudeney's Haberdasher's Puzzle	92
5.	Reversible Transformation of Figures	97
6.	Tessellability and Shapes of Trunks	107
7.	Miscellaneous Properties of Uni-Trunk Holders	112

8. Conway Polygons	118
9. Art Based on Reversible Pairs of Figures	125
Appendix 4.4.1	127
Appendix 4.5.1	128
Appendix 4.7.1	130
Appendix 4.8.1	131
Appendix 4.8.2	137
References	141
Chapter 5 Platonic Solids	143
1. The Platonic Solids	143
2. Euler's Formula for Polyhedra	145
3. Plato's Image for Platonic Solids	147
4. Duality Between a Pair of Polyhedra	150
5. The Cycle of Platonic Solids	151
Appendix 5.2.1	153
Appendix 5.3.1	156
References	158
Chapter 6 Cross-Sections of Polyhedra	159
1. CT Scanners and 3-D Copiers	159
2. Cross-Sections of a Cube	161
3. Universal Tilers	165
4. Mitsubishi and the Star of David	166
References	174
Chapter 7 Symmetry of Platonic Solids	175
1. Diagonal Weights	175
2. A Soccer Ball and a Fullerene	180
3. The Platonic Solids through Reflecting Mirrors	183
Appendix 7.1.1	189
Appendix 7.1.2	189
Appendix 7.1.3	190
References	191
Chapter 8 Double Duty Solids	193
1. TetraPak®	193
2. Double Duty Solids	197
3. Finding More Double Duty Solids	200
4. Eleven Types of e-Nets of a Cube	205
5. References	212
Chapter 9 Nets of Small Solids with Minimum Perimeter Lengths	213
1. Minimum Steiner Tree Problem	213
2. Melzak's Algorithm	219
3. Minimum Perimeter Nets for a Regular Octahedron $\bowtie$	221
4. Minimum Perimeter Net for a Cube $\mathcal{C}$	223
5. Soap Experiments for Minimum Steiner Trees	225
6. Soap Experiments for Nets with Minimum Perimeter Lengths for Small Solids	227

Appendix 9.1.1	231
Appendix 9.2.1	232
References	233
Chapter 10 Tessellation Polyhedra	235
1. Regular Faced Polyhedra	235
2. Tessellation Polyhedra	238
3. Art	245
Appendix 10.2.1	254
References	255
Chapter 11 Universal Measuring Boxes	257
1. How to Measure Water Using a 6 dL Box	257
2. Changing the Shape of the Box	262
3. Orthogonal Measuring Boxes	265
4. An Application of the Ternary System	272
5. Non-Orthogonal Universal Measuring Boxes	277
Appendix 11.4.1	282
References	283
Chapter 12 Wrapping a Box	285
1. Wrapping Technique of the Experts	285
2. Point Symmetric Skew Wrapping	286
3. Patchwork Surgery	290
4. Investigating Efficient Wrappings	295
5. Six Remarkable Wrappings Obtained by Patchwork Surgery	300
6. The Most Saving But Mummy-Like Wrapping	303
References	308
Chapter 13 Bees, Pomegranates and Parallelhedra	309
1. How to Pack Cans Efficiently	309
2. Honeycombs and Pomegranate Seeds	312
3. Applications of Sphere Packing	315
4. Rhombic Dodecahedron	319
5. Truncated Octahedron	320
6. Lattices and Voronoi Domains	323
7. Parallelhedra	329
References	333
Chapter 14 Reversible Polyhedra	335
1. A Pig to a Ham, a Fox to a Snake, and Panda Magic	335
2. Transformation of a Pig to Ham	339
3. How to Make a Fox-Snake Solid	341
4. Make a Fox-Snake Solid with Paper	347
5. Making Panda Magic!	349
6. How to Make Reversible Polyhedra	350
7. Who Will Come Out of the U.F.O.?	352
8. P_1 -lattices and (P, Q)-Chimera Superimpositions	354
9. P_1 -Reversible Pairs Among Canonical Parallelhedra	359

10. Reversibility for Parallelohedra and the Double-Reversal-Plates Method	367
References	372
Chapter 15 Elements of Polygons and Polyhedra	373
1. The Power of Triangulation	373
2. Element Sets and Element Numbers	378
3. Hilbert's Third Problem and Dehn Theorem	382
4. An Element Set and the Element Number for Platonic Solids	388
5. Another Element Set for the Platonic Solids	390
Appendix 15.4.1	395
References	398
Chapter 16 The Pentadron	399
1. Atoms for Parallelohedra	399
2. Another Way to Find a Pair of Pentadra	406
3. The Chart for Composing Five Parallelohedra with Pentadra	407
4. Pentadral Complices	410
5. A Bridge Between Platonic Solids and Parallelohedra	412
6. Art and Design	413
Appendix 16.1.1	414
References	416
Figure Permissions	417
© J Art 2015	418
Copyright	419
Endnote	420
Index	421
Symbols	424

About this Book

The characteristics of this book are the following:

- (a) The theorems and formulas that are presented encourage the reader to discover for him or herself (heuristic approach).
- (b) Most of the theorems are presented with a story of how the authors were inspired and came up with the idea for the theorem.
- (c) This book introduces not only key results and tools in each topic, but also many original results obtained by the authors. This is done through casual conversation between Gen and Kyuta. Gen (a mathematician), visits Kyuta and teaches him to appreciate the excitement of creating theorems together with art using mathematics. Most of the topics are original and have not been introduced in other books.
- (d) The target readers are undergraduate students; however, this book is self-contained and only requires knowledge at the high school level.
- (e) Many of the latest unique and beautiful results in geometry (in particular on polygons and polyhedra) and the dynamism of mathematical research history may also captivate adults and even researchers.
- (f) Readers can create their own works of art by applying the theorems presented in this book, as the procedures are explained explicitly.
- (g) The book is illustrated with color photographs of works of art and design that have been created using the theorems and procedures in the book.

About the Authors

Jin Akiyama is a mathematician at heart. Currently, he is a fellow of the European Academy of Science, the founding editor of the journal *Graphs and Combinatorics*, and the director of the Research Institute of Math Education at Tokyo University of Science. He is particularly interested in graph theory and discrete and solid geometry and has published many papers in these fields. Aside from his research, he is best known for popularizing mathematics, first in Japan and then in other parts of the globe: his lecture series was broadcast on NHK (Japan's only public broadcaster) television and radio from 1991 to 2013. He was a founding member of the Organizing Committee of the UNESCO-sponsored traveling exhibition "Experiencing Mathematics", which debuted in Denmark in 2004. In 2013, he built a hands-on mathematics museum called Akiyama's Math Experience Plaza in Tokyo. He has authored and co-authored more than 100 books, including *Factors and Factorizations of Graphs* (jointly with Mikio Kano, Lect. Notes Math., Springer, 2011) and *A Day's Adventure in Math Wonderland* (jointly with Mari-Jo Ruiz, World Scientific, 2008), which has been translated into nine languages.

Kiyoko Matsunaga is a science writer for mathematics. She has contributed not only through Japanese-language books, but also in NHK TV programs on mathematics, including *Math, I Like It* for elementary students, *Math Time Travel* for junior and senior high school students, and *The Joy of Math* and *Math Wonderland* for the broader public.