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Numerical Solutions of Partial Differential Equations

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Foreword

This book contains an expanded and smoothed version of lecture notes delivered by the authors at the Advanced School on *Numerical Solutions of Partial Differential Equations: New Trends and Applications*, which took place at the Centre de Recerca Matemàtica (CRM) in Bellaterra (Barcelona) from November 15th to 22nd, 2007.

The book has three parts. The first part, by Silvia Bertoluzza and Silvia Falletta, is devoted to the use of wavelets to derive some new approaches in the numerical solution of PDEs, showing in particular how the possibility of writing equivalent norms for the scale of Besov spaces allows to write down some new methods. The second part, by Giovanni Russo, provides an overview of the modern finite-volume and finite-difference shock-capturing schemes for systems of conservation and balance laws, with emphasis in giving a unified view of such schemes by identifying the essential aspects of their construction. In the last part Chi-Wang Shu gives a general introduction to the discontinuous Galerkin methods for solving some classes of PDEs, discussing cell entropy inequalities, nonlinear stability and error estimates.

The school that originated these notes was born with the objective of providing an opportunity for PhD students, recent PhD doctorates and researchers in general in fields of applied mathematics and engineering to catch up with important developments in the fields and/or to get in touch with state-of-the-art numerical techniques that are not covered in usual courses at graduate level.

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