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Nonlocal Elliptic Operators

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Preface

Noncommutative geometry, which can rightfully claim the role of a philosophy in mathematical studies, undertakes to replace good old notions of classical geometry (such as manifolds, vector bundles, metrics, differentiable structures, etc.) by their abstract operator-algebraic analogs and then to study the latter by methods of the theory of operator algebras. At first sight, this pursuit of maximum possible generality harbors the danger of completely forgetting the classical beginnings, so that not only the answers but also the questions would defy stating in traditional terms. Noncommutative geometry itself would become not only a method but also the main subject of investigation according to the capacious but not too practical formula: "Know thyself." Fortunately, this is not completely true (or even is completely untrue) in reality: there are numerous problems that are quite classical in their statement (or at least admit an equivalent classical statement) but can be solved only in the framework of noncommutative geometry. One of such problems is the subject of the present book.

The classical elliptic theory developed in the well-known work of Atiyah and Singer on the index problem relates an analytic invariant of an elliptic pseudodifferential operator on a smooth compact manifold, namely, its index, to topological invariants of the manifold itself. The index problem for *nonlocal* (and hence nonpseudodifferential) elliptic operators is much more complicated and requires the use of substantially more powerful methods than those used in the classical case. It should be noted that although elliptic theory (more precisely, its analytic branch) for nonlocal operators has been studied sufficiently well, meaningful results concerning the index problem for nonlocal elliptic operators have until recently been rather spare. It is only very recently that several important results (some of which are due to the authors) have appeared, suggesting that the solution of this problem is eventually within reach. Therefore, there is a need to gather together the facts already known on the topic.

That is why the present book has been written. Methods of K-theory of operator algebras and noncommutative geometry are used here to solve the index problem for nonlocal elliptic operators associated with a countable group of diffeomorphisms of a manifold.

Furthermore, to make the presentation self-contained and hence the book understandable for readers with standard university education in mathematics, we have decided to include the Appendix, which contains some material used in noncommutative elliptic theory, namely, C^*-algebras and their K-theory as well as basics of the theory of cyclic homology and cohomology.

The main results contained in the book were obtained during the authors' stay as guests researchers at Institut für Analysis, Leibniz Universität Hannover (Hannover, Germany).

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We are grateful to Professor B.-W. Schulze of Potsdam University, who suggested publishing our recent results on the noncommutative theory of differential equations with Birkhäuser.

Finally, we especially wish to thank Professor G. V. Rozenblyum of Göteborg University, Sweden, who was the first to acquaint us with the beautiful world of noncommutative geometry.

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