

Part II

**Abelian Integrals and
Applications to the Weak
Hilbert's 16th Problem**

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Dedicated to Professor Zhifen Zhang on the occasion of her 80th birthday

Preface

The second part of Hilbert's 16th problem, asking for the maximum $H(n)$ of the numbers of limit cycles and their relative positions for all planar polynomial differential systems of degree n , is still open even for the quadratic case ($n = 2$).

A weak form of this problem, proposed by Arnold, asking for the maximum $Z(m, n)$ of the numbers of isolated zeros of Abelian integrals of all polynomial 1-forms of degree n over algebraic ovals of degree m , is also extremely hard to grasp. The number $\tilde{Z}(n) = Z(n + 1, n)$ can be chosen as a lower bound of $H(n)$; so far only $\tilde{Z}(2) = 2$ has been proved.

These lecture notes are devoted to the introduction of some basic concepts and methods in the study of Abelian integrals and applications to the weak Hilbert's 16th problem. In Chapter 1 we briefly introduce Hilbert's 16th problem and its weak form. In Chapter 2 we explain the relation between the study of Abelian integrals and the study of limit cycles. In Chapter 3 we use several methods to study the number of zeros of the Abelian integrals associated with perturbations of the Bogdanov–Takens system. At last, in Chapter 4 we introduce a proof of $\tilde{Z}(2) = 2$, the method of the proof is unified for all regions of the parameter space.

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