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Limit Cycles of Differential Equations

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Foreword

This book contains two sets of revised and augmented notes prepared for the Advanced Course on Limit Cycles and Differential Equations given at the Centre de Recerca Matemàtica in June 2006, as part of its year-long research programme on Hilbert's 16th problem. The common goal of the two sets of notes is to help young mathematicians enter a very active area of research lying on the borderline between dynamical systems, analysis and applications.

The first part of the book, by Colin Christopher, considers some of the topics which surround the Poincaré center-focus problem for polynomial systems, a subject closely tied with the integrability of polynomial systems. The second part, by Chengzhi Li, is devoted to the introduction of some basic concepts and methods in the study of Abelian integrals and applications to the weak Hilbert's 16th problem.

Besides our indebtedness to the Centre de Recerca Matemàtica, thanks are due to Jaume Llibre and Armengol Gasull, the course co-ordinators, for giving us this challenging but rewarding opportunity and for providing such a pleasant environment during the programme.

Contents

I	Around the Center-Focus Problem	
	<i>Colin Christopher</i>	1
	Preface	3
1	Centers and Limit Cycles	5
1.1	Outline of the Center-Focus Problem	5
1.2	Calculating the Conditions for a Center	9
1.3	Bifurcation of Limit Cycles from Centers	10
2	Darboux Integrability	17
2.1	Invariant Algebraic Curves	17
2.2	The Darboux Method	18
2.3	Multiple Curves and Exponential Factors	21
3	Liouvillian Integrability	25
3.1	Differential Fields and Liouvillian Extensions	25
3.2	Proof of Singer's Theorem	26
3.3	Riccati equations	29
4	Symmetry	33
4.1	Algebraic Symmetries	33
4.2	Centers for analytic Liénard equations	34
4.3	Centers for polynomial Liénard equations	37
5	Cherkas' Systems	41
6	Monodromy	49
6.1	Some Basic Examples	49
6.2	The Model Problem	50
6.3	Applying Monodromy to the Model Problem	51
7	The Tangential Center-Focus Problem	55
7.1	Preliminaries	56
7.2	Generic Hamiltonians	57
7.3	Relative exactness	59
8	Monodromy of Hyperelliptic Abelian Integrals	63
8.1	Some Group Theory	64
8.2	Monodromy groups of polynomials	65
8.3	Proof of the theorem	67

9	Holonomy and the Lotka–Volterra System	71
9.1	The monodromy group of a separatrix	72
9.2	Integrable points in Lotka–Volterra systems	73
10	Other Approaches	79
10.1	Finding components of the center variety	79
10.2	Extending Centers	80
10.3	An Experimental Approach	82
	Bibliography	85
II	Abelian Integrals and Applications to the Weak Hilbert’s 16th Problem	
	<i>Chengzhi Li</i>	91
	Preface	93
1	Hilbert’s 16th Problem and Its Weak Form	95
1.1	Hilbert’s 16th Problem	95
1.2	Weak Hilbert’s 16th Problem	99
2	Abelian Integrals and Limit Cycles	111
2.1	Poincaré–Pontryagin Theorem	111
2.2	Higher Order Approximations	116
2.3	The Integrable and Non-Hamiltonian Case	120
2.4	The Study of the Period Function	122
3	Estimate of the Number of Zeros of Abelian Integrals	127
3.1	The Method Based on the Picard–Fuchs Equation	127
3.2	A Direct Method	130
3.3	The Method Based on the Argument Principle	133
3.4	The Averaging Method	138
4	A Unified Proof of the Weak Hilbert’s 16th Problem for $n=2$	143
4.1	Preliminaries and the Centroid Curve	143
4.2	Basic Lemmas and the Geometric Proof of the Result	145
4.3	The Picard–Fuchs Equation and the Riccati Equation	149
4.4	Outline of the Proofs of the Basic Lemmas	155
4.5	Proof of Theorem 4.6	156
	Bibliography	159