

Part I

**Around the
Center-Focus Problem**

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Preface

My aim in these notes is to consider some of the topics which surround the Poincaré center-focus problem for polynomial systems. That is, given a polynomial system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

with a critical point whose linearization gives a center, under what conditions can we conclude that the point is a center for the nonlinear system?

Clearly, the subject is closely tied with what mechanisms underlie the local integrability of polynomial systems, since the existence of a center implies the existence of a local analytic first integral.

Because these systems are defined algebraically, we expect these mechanisms to be algebraic too, in some sense. This indeed seems to be the case, but the situation is far from being understood except for a growing number of explicit examples.

The choice of topics covered in these notes is very much a personal one, being in the main problems that I have been involved in myself or found interesting. Unfortunately, this has meant that there is much that is missing from this presentation which I felt less competent to comment on. In particular, very little is said on the many detailed analyses of particular systems, nor on the more far-reaching work on holomorphic foliations.

The first part of the notes considers the two main mechanisms known to produce centers in polynomial systems, namely Darboux integrability and algebraic symmetries. The second part considers several topics loosely associated with the idea of monodromy. Though diverse, they share a common theme of teasing out concrete global information from trying to extend the known local behavior, surely one of the most beguiling aspects of the center-focus problem.