Frontiers in Mathematics

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Rings, Modules, and the Total

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Preface

This book is an introductory text to the theory of rings and modules with special emphasis on decomposition properties. The total is a concept that is as fundamental as that of the radical, and has not appeared in text books before. Except for the most basic concepts and facts of ring and module theory, the book is self–contained, and proofs are included even of rather well–known results. Yet, the book reaches into the frontier of the subject.

What are the best elements in a ring R (always with 1)? Obviously, these are the invertible elements that are the divisors of 1. We call the divisors of an idempotent $0 \neq e = e^2$ *partially invertible*, so r is partially invertible if there exists $s \in R$ such that $e = sr$. In this book we intend to show that the partially invertible elements are the "second best" elements in a ring. The set of elements of R that are not partially invertible we denote by $\mathrm{Tot}(R)$ and call it the *total* of R.

These notions are not restricted to rings. If Mod–R is the category of all unitary right R–modules $A, B, C, M, W, \ldots$, then $f \in \mathrm{Hom}_R(M, W)$ is called partially invertible if there exists $g \in \mathrm{Hom}_R(W, M)$ such that

$$\mathrm{End}(M) \ni e := gf = e^2 \neq 0,$$

which again means that f is a divisor of a non–zero idempotent. This is equivalent to the fact that there exist non–zero direct summands $A \subseteq^{\oplus} M$, $B \subseteq^{\oplus} W$ such that f induces an isomorphism

$$A \ni a \mapsto f(a) \in B.$$

Homomorphisms $f \in \mathrm{Hom}_R(M, W)$ that are not partially invertible are called *total non–isomorphisms*. The set of total non–isomorphisms on M to W is called the *total on M to W* and denoted by $\mathrm{Tot}(M, W)$. These notions were defined by the first author of this book in 1982 and studied in his seminar in Munich. They generalize special concepts used in the theory of modules with local endomorphism rings. The first publication on the total in this generality was a PhD thesis by W. Schneider ([27]). In the following years several authors dealt with the concepts or used them (Beidar [4], Beidar and Kasch [5], [6], Beidar and Wiegand [7], Kasch [16], [17], [18], [19], [20], Kasch and Schneider [21], [22], [27], Zelmanowitz [28], Zöllner [29]). By now there are numerous results and most of them are covered in this book.

In Chapter II we derive the fundamental properties. The total is in general not additively closed but it has the following multiplicative closure property:

If $X, Y \in \text{Mod–}R$ *and* $h \in \text{Hom}_R(X, M)$, $k \in \text{Hom}_R(W, Y)$, *then* $k \,\text{Tot}(M, W) h \subseteq \text{Tot}(X, Y)$.

A set of maps with this property is called a *semi–ideal* in Mod–R. Thus, for a ring R,

$$x, y \in R \Rightarrow x \,\text{Tot}(R) y \subseteq \text{Tot}(R).$$

Although $\text{Tot}(M, W)$ is not additively closed, it has certain additive closure properties. We introduce (Definition 2.2)

$$\text{Rad}(M, W) := \{f \in \text{Hom}_R(M, W) \mid \forall\, g \in \text{Hom}_R(W, M), fg \in \text{Rad}(S)\}.$$

The radical $\text{Rad}(M, W)$ contains the usual radicals of $\text{Hom}_R(M.W)$ both as a left $\text{End}_R(M)$–module and as a right $\text{End}_R(W)$–module. Also, for a ring R identified naturally with $\text{Hom}_R(R_R, R_R)$ we have $\text{Rad}(R_R, R_R) = \text{Rad}(R)$. It is now true that

$$\text{Rad}(M, W) + \text{Tot}(M, W) = \text{Tot}(M, W),$$

which also implies that $\text{Rad}(M, W) \subseteq \text{Tot}(M, W)$. In particular, in the ring case, $\text{Rad}(R) \subseteq \text{Tot}(R)$.

Not only is it true that $\text{Rad}(M, W) \subseteq \text{Tot}(M, W)$, but also the *singular submodule* $\Delta(M, W)$ (the set of all $f \in \text{Hom}_R(M, W)$ with large kernel) and the *co–singular submodule* $\nabla(M, W)$ (the set of all $f \in \text{Hom}_R(M, W)$ with small image) are contained in $\text{Tot}(M, W)$. A natural question arises: For which M and W is it true that $\text{Rad}(M, W) = \text{Tot}(M, W)$ or $\Delta(M, W) = \text{Tot}(M, W)$ or $\nabla(M, W) = \text{Tot}(M, W)$? Since Rad, Δ, and ∇ are additively closed, in the case that any one of the above equalities holds, the total is also additively closed. For the question of equality we provide some interesting answers in Chapter III (see III.Theorem 2.2, III.Corollary 2.5). Also exchange properties for modules imply good additive properties for the total.

In Chapter IV we present a difficult part of algebra. If a module M has a decomposition

$$M = \bigoplus_{i \in I} M_i$$

where all endomorphism rings $S_i := \text{End}(M_i)$, $i \in I$, are local rings, then the decomposition is called an *LE–decomposition.*

Modules with LE–decompositions were studied by many algebraists and success in the subject required hard work. By now there exists a satisfactory theory of these modules. We present the main part of this theory. If $S := \text{End}(M)$, then the existence of an LE–decomposition implies that $\text{Tot}(S)$ is an ideal in S, hence $S/\text{Tot}(S)$ is again a ring.

First Main Theorem (IV.3.3). *Assume that* M *is a module with an LE–decomposition, and set* $S := \text{End}(M)$. *Then the quotient ring* $S/\text{Tot}(S)$ *is isomorphic to a product of endomorphism rings of vector spaces over division rings.*

Examples (IV.Example 2.5) show that for LE–decompositions $\mathrm{Rad}(S)$ need not be equal to $\mathrm{Tot}(S)$. Hence for the First Main Theorem, the total is truly the essential notion. But if $\mathrm{Rad}(S) = \mathrm{Tot}(S)$, then there are further "very good" properties.

Second Main Theorem (IV.4.1). *Assume that M is a module that has an LE–decomposition, and set $S := \mathrm{End}(M)$. Then the following statements are equivalent.*

1) $\mathrm{Rad}(S) = \mathrm{Tot}(S)$.
2) *Every LE–decomposition of M complements direct summands.*
3) *Any two LE–decompositions of M have the replacement property.*
4) *If $M = \bigoplus_{i \in I} M_i$ is an LE–decomposition, then the family $\{M_i \mid i \in I\}$ is a local semi–t–nilpotent family.*

A direct decomposition $M = \bigoplus_{i \in I} M_i$ *complements direct summands* if for every direct summand A of M there is a set of indices $J \subseteq I$ such that

$$M = A \oplus \bigoplus_{j \in J} M_j.$$

Two LE–decompositions

$$M = \bigoplus_{i \in I} M_i = \bigoplus_{j \in J} N_j$$

satisfy the *replacement property* if *for each subset $I_0 \subseteq I$ there exists a subset $J_0 \subseteq J$ such that*

$$M = \left(\bigoplus_{i \in I \setminus I_0} M_i\right) \oplus \left(\bigoplus_{j \in J_0} N_j\right).$$

A family of LE–modules (= modules with local endomorphism ring) $\{M_i \mid i \in I\}$ is called *locally semi–t–nilpotent* if and only if for every infinite sequence of pairwise different elements

$$i_1, i_2, i_3, \ldots \in I$$

and for every sequence of homomorphisms

$$f_1, f_2, f_3, \ldots \text{ with } f_j \in \mathrm{Tot}(M_{i_j}, M_{i_{j+1}})$$

and for every $x \in M_{i_1}$, there exists $n \in \mathbb{N}$ such that

$$f_n f_{n-1} f_{n-2} \cdots f_2 f_1(x) = 0.$$

Who was involved in formulating and proving this theorem? It is difficult to give a precise answer including all progress and names. We believe that Harada developed a major part of the theory of LE–decompositions. Therefore we only cite one paper by Harada ([12]) that will serve as a source of further information. After Harada (1974) proofs were improved by several authors (e.g. [29]).

In the last chapter we consider completely decomposable Abelian groups that constitute the simplest interesting class of torsion–free Abelian groups. After providing the necessary background for the novice in this subject, we compute the total of the endomorphism ring in a special case (V.Theorem 2.1) and describe a recursive method for determining which maps in the endomorphism ring of a completely decomposable group belongs to the total.

In a final section we discuss torsion–free Abelian groups of finite rank in general. These groups have very ill–behaved direct decompositions (V.Example 3.1) but by moving to an associated category one obtains LE–decompositions and the total of an endomorphism ring in this category is just the radical (V.Corollary 3.16, V.Theorem 3.17).

We hope that this monograph will serve as an introduction to the total and form the basis for further progress.