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Relaxation and Decomposition Methods for Mixed Integer Nonlinear Programming

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Preface

Nonlinear optimization problems containing both continuous and discrete variables are called mixed integer nonlinear programs (MINLP). Such problems arise in many fields, such as process industry, engineering design, communications, and finance.

There is currently a huge gap between MINLP and mixed integer linear programming (MIP) solver technology. With a modern state-of-the-art MIP solver it is possible to solve models with millions of variables and constraints, whereas the dimension of solvable MINLPs is often limited by a number that is smaller by three or four orders of magnitude. It is theoretically possible to approximate a general MINLP by a MIP with arbitrary precision. However, good MIP approximations are usually much larger than the original problem. Moreover, the approximation of nonlinear functions by piecewise linear functions can be difficult and time-consuming.

In this book relaxation and decomposition methods for solving nonconvex structured MINLPs are proposed. In particular, a generic *branch-cut-and-price* (BCP) framework for MINLP is presented. BCP is the underlying concept in almost all modern MIP solvers. Providing a powerful decomposition framework for both sequential and parallel solvers, it made the success of the current MIP technology possible. So far generic BCP frameworks have been developed only for MIP, for example, COIN/BCP (IBM, 2003) and ABACUS (OREAS GmbH, 1999). In order to generalize MIP-BCP to MINLP-BCP, the following points have to be taken into account:

- A given (sparse) MINLP is reformulated as a block-separable program with linear coupling constraints. The block structure makes it possible to generate Lagrangian cuts and to apply Lagrangian heuristics.
- In order to facilitate the generation of polyhedral relaxations, nonlinear convex relaxations are constructed.
- The MINLP separation and pricing subproblems for generating cuts and columns are solved with specialized MINLP solvers.
- Solution candidates are computed via MINLP heuristics by using an NLP solver.

I started to work on these tasks in 1996 when I implemented a branch-and-bound algorithm for solving polynomial programs based on multivariate Bézier polynomials (Nowak, 1996). Since polynomial programs can be reformulated as all-quadratic programs, I got interested in semidefinite programming relaxations. At this time I learned from Werner Römisch and Krzysztof Kiwiel about Lagrangian decomposition in the context of stochastic programming. Motivated by both approaches, I started in 2000 to implement an object oriented library, called LaGO (Lagrangian Global Optimizer), for solving nonconvex mixed-integer all-quadratic programs (MIQQPs) based on Lagrangian decomposition and semidefinite relaxation. From 2001 until 2003, LaGO was extended in a project funded by the German Science Foundation to solve nonconvex MINLPs.

This book documents many of the theoretical and algorithmic advances that made the development of LaGO possible and that give suggestions for further improvements. The most important contributions are:

- Several estimates on the duality gap (Sections 3.4, 3.5 and 5.4).
- A new column generation method for generating polyhedral inner and outer approximations of general MINLPs (Section 4.3).
- A new decomposition-based method for solving the dual of general MIQQPs through eigenvalue computation (Section 5.3).
- A new lower bounding method for multivariate polynomials over simplices based on Bernstein–Bézier representations (Section 6.2).
- A new polynomial underestimator for general nonconvex multivariate black-box functions (Section 6.5).
- New locally exact cuts based on interval arithmetic (Section 7.1.3).
- Decomposition-based lower bounds and box-reduction techniques for MINLPs (Sections 7.3 and 7.4).
- Optimality cuts and global optimality criteria for quadratically constrained quadratic programs (QQPs) based on a new strong duality result (Chapter 8).
- A new adaptive method for simultaneously generating discretizations and computing relaxations of infinite dimensional MINLPs (Chapter 9).
- New deformation heuristics for MaxCut and MINLP (Chapter 11) based on convex relaxations.
- Rounding and partitioning heuristics for MINLP (Sections 12.1 and 12.2).
- A Lagrangian heuristic for MINLP (Section 12.4).
- The first BCP algorithm for general MINLPs (Chapter 13).
- The first finiteness proof for QQP branch-and-bound methods that use optimality cuts (Section 13.2).

- A tool for automatically generating a block-separable reformulation of a black-box MINLP (Sections 2.3.2 and 14.4.1).

The use of relaxation-based methods for solving practically relevant large-scale MINLPs is quite new, and the integration of the two well established areas, nonlinear and mixed integer optimization, does not belong to the "traditional" operation research areas yet. However, according to a recent paper on future perspectives of optimization (Grossmann and Biegler, 2002) this can change in the future.

This monograph can be used both as a research text and as an introduction into MINLP. It is subdivided into two parts. The first part provides some basic concepts and the second part is devoted to solution algorithms.

Chapters 1 and 2 give an introduction into structured MINLPs and discuss various ways of reformulating a MINLP to be block-separable. Chapters 3, 4, 5, 6, 7 are devoted to theory and computational methods for generating Lagrangian and convex relaxations. Chapters 8 and 9 present global optimality cuts and a new method for refining discretizations of infinite dimensional MINLPs.

Chapter 10 gives an overview on existing global optimization methods. Chapters 11 and 12 describe deformation, rounding-and-partitioning and Lagrangian heuristics. Chapter 13 presents branch-cut-and-price algorithms for general MINLPs.

Chapter 14 contains a short description of the MINLP solver LaGO. Appendices A and B discuss future perspectives on MINLP and describe the MINLP instances used in the numerical experiments.

Berlin, January 2005 Ivo Nowak

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Notation

∇	gradient column vector.
∇^2	Hessian.
∇_x	gradient with respect to x.
$\partial f(x)$	subdifferential of f at x.
x^T	the transpose of the vector x.
x_I	sub-vector defined by $(x_i)_{i \in I}$.
$f_I(x)$	sub-function defined by $(f_i(x))_{i \in I}$.
$A_{I,J}$	sub-matrix defined by $(a_{ij})_{i \in I, j \in J}$.
$\|x\|$	Euclidian norm of the vector x.
$\langle x, y \rangle$	scalar product of vectors x and y.
$\langle A, B \rangle$	inner product of matrices A and B defined by trace AB.
$f(x; t)$	function depending on a variable x and a parameter t.
$\bar{f}(x; z)$	linear approximation to f at z evaluated at x.
$\|I\|$	cardinality of the index set I.
μ	dual point (Lagrangian multiplier).
$L(x; \mu)$	Lagrangian function.
$D(\mu)$	dual function.
$\mathcal{M}$	Lagrangian multiplier set.
$\mathbb{B}^n(\rho, x)$	n-dimensional ball with center x and radius ρ.
$\mathbb{B}^n(\rho)$	$\mathbb{B}^n(\rho, 0)$.
$\mathbb{B}(n)$	$\mathbb{B}^n(\sqrt{n}, 0)$.
$\mathbb{S}^n$	n-sphere.
Δ_n	standard simplex in $\mathbb{R}^n$.
$\lambda_1(A)$	smallest eigenvalue of a matrix A.
$A \succcurlyeq B$	Loewner order of symmetric matrices defined by $A \succcurlyeq B \Leftrightarrow A - B$ is positive semi-definite.
$\operatorname{conv}(S)$	convex hull of the set S.
$\operatorname{vert}(S)$	extreme points of the set S.
$\operatorname{int} S$	interior of the set S.
$\breve{S}$	convex relaxation of the set S.
$\hat{S}$	polyhedral outer approximation of the set S.
$\check{S}$	polyhedral inner approximation of the set S.

$\imath_\Omega(x)$	characteristic function.
$\mathrm{val}(P)$	optimal value of the optimization problem (P).
$\mathrm{sol}(P)$	solution set of the optimization problem (P).