Elliott H. Lieb
Robert Seiringer
Jan Philip Solovej
Jakob Yngvason

The Mathematics of the Bose Gas and its Condensation

Birkhäuser Verlag
Basel · Boston · Berlin

Authors:

Elliott H. Lieb
Departments of Mathematics and Physics
Princeton University
Jadwin Hall, P.O. Box 708
Princeton, NJ 08544
USA
lieb@math.princeton.edu

Robert Seiringer
Department of Physics
Princeton University
Jadwin Hall, P.O. Box 708
Princeton, NJ 08544
USA
rseiring@math.princeton.edu

Jan Philip Solovej
Department of Mathematics
University of Copenhagen
Universitetsparken 5
2100 Copenhagen
Denmark
solovej@math.ku.dk

Jakob Yngvason
Institut für Theoretische Physik
Universität Wien
Boltzmanngasse 5
1090 Wien
Austria
yngvason@thor.thp.univie.ac.at

2000 Mathematical Subject Classification 82Bxx

A CIP catalogue record for this book is available from the
Library of Congress, Washington D.C., USA

Bibliografische Information Der Deutschen Bibliothek
Die Deutsche Bibliothek verzeichnet diese Publikation in der Deutschen Nationalbibliografie;
detaillierte bibliografische Daten sind im Internet über <http://dnb.ddb.de> abrufbar.

ISBN 3-7643-7336-9 Birkhäuser Verlag, Basel – Boston – Berlin

Published by Birkhäuser Verlag, P.O. Box 133, CH-4010 Basel, Switzerland
Part of Springer Science+Business Media
Cover design: Micha Lotrovsky, CH-4106 Therwil, Switzerland
Printed on acid-free paper produced from chlorine-free pulp. TCF ∞
Printed in Germany
ISBN-10: 3-7643-7336-9
ISBN-13: 978-3-7643-7336-8

9 8 7 6 5 4 3 2 1 www.birkhauser.ch

Contents

Preface

The mathematical study of the Bose gas goes back to the first quarter of the twentieth century, with the invention of quantum mechanics. The name refers to the Indian physicist S.N. Bose who realized in 1924 that the statistics governing photons (essentially invented by Max Planck in 1900) is determined (using modern terminology) by restricting the physical Hilbert space to be the symmetric tensor product of single photon states. Shortly afterwards, Einstein applied this idea to massive particles, such as a gas of atoms, and discovered the phenomenon that we now call Bose-Einstein condensation. At that time this was viewed as a mathematical curiosity with little experimental interest, however.

The peculiar properties of liquid Helium (first liquefied by Kammerlingh Onnes in 1908) were eventually viewed as an experimental realization of Bose-Einstein statistics applied to Helium atoms. The unresolved mathematical problem was that the atoms in liquid Helium are far from the kind of non-interacting particles envisaged in Einstein's theory, and the question that needed to be resolved was whether Bose-Einstein condensation really takes place in a strongly interacting system — or even in a weakly interacting system.

That question is still with us, three quarters of a century later!

The first systematic and semi-rigorous mathematical treatment of the problem was due to Bogoliubov in 1947, but that theory, while intuitively appealing and undoubtedly correct in many aspects, has major gaps and some flaws. The 1950's and 1960's brought a renewed flurry of interest in the question, but while theoretical intuition benefited hugely from this activity the mathematical structure did not significantly improve.

The subject was largely quiescent until the 1990's when experiments on low density (and, therefore, weakly interacting instead of strongly interacting, as in the case of liquid Helium) gases showed for the first time an unambiguous manifestation of Bose-Einstein condensation. This created an explosion of activity in the physics community as can be seen from the web site
`http://bec01.phy.georgiasouthern.edu/bec.html/bibliography.html`,
which contains a bibliography of several thousand papers related to BEC written in the last 10 years.

At more or less the same time some progress was made in obtaining rigorous mathematical proofs of some of the properties proposed in the 50's and 60's. A general proof of Bose-Einstein condensation for interacting gases still eludes us, but we are now in a much stronger position to attack this problem rigorously. These notes, which are an extension of our 2004 Oberwolfach course, summarize some rigorous results that have been obtained by us in the past decade. Most of them are about the ground state energy in various models and dimensions, but we do have a few results about the occurrence (and non-occurrence) of condensation.

This pedagogical summary has several antecedents. It has grown organically as new results emerged. The first one was [LY3], followed by [LSeY3], [Se2], [L7], [LSeY4], [LSSY], and [LSSY2]. Apart from this stream, there was another pedagogical survey going back to the 60's [L3] that dealt with Bogoliubov theory and other things. Some of that material is reproduced in Appendices A and B.

There is, of course, a large body of rigorous work by other people on various aspects of BEC that was not covered in the Oberwolfach course and is not mentioned in these notes. The subject can be approached from many angles and our aim was not to give a complete overview of the subject but to focus on themes where we have been able to make some contributions. The recent Physics Reports article [ZB] on the Bogoliubov model is a good source of references to some other approaches and results. There exist also several reviews, e.g., [DGPS, ISW, Leg, C, Yu] and even monographs [PS, PiSt2] on the fascinating physics of the Bose gas and its condensation.

Acknowledgments

Our thanks go to Letizzia Wastavino for producing the two figures and to Michael Aizenman and Kai Schnee for permission to include the material in Chapters 11 and 9, respectively. We are also grateful to Daniel Ueltschi, Bruno Nachtergaele and Valentin Zagrebnov for very useful comments on various parts of these notes. Finally, we thank the Oberwolfach Institute for the opportunity to give this course and we thank the participants for pointing out numerous typos in a previous version of the manuscript. The work was supported in part by US NSF grants PHY 0139984-A01 (EHL), PHY 0353181 (RS) and DMS-0111298 (JPS); by an A.P. Sloan Fellowship (RS); by EU grant HPRN-CT-2002-00277 (JPS and JY); by the Alexander von Humboldt Foundation (EHL); by FWF grant P17176-N02 (JY); by MaPhySto – A Network in Mathematical Physics and Stochastics funded by The Danish National Research Foundation (JPS), and by grants from the Danish research council (JPS).