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Quadrature Domains and Their Applications

The Harold S. Shapiro Anniversary Volume

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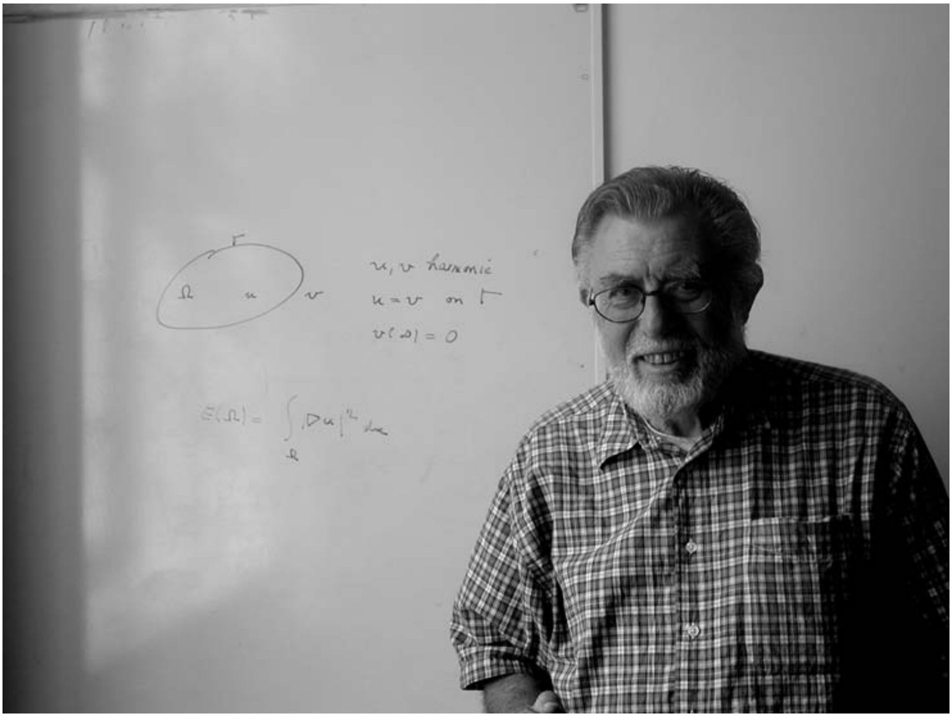
Preface

This book is an expanded version of talks and contributed papers presented at a conference held at the University of California at Santa Barbara in March 2003, to celebrate the 75th birthday of Professor Harold S. Shapiro. The main theme of the conference was Quadrature Domains and Their Applications.



Harold S. Shapiro

The idea of having Gaussian type quadratures, on a fixed domain and for the full class of integrable analytic or harmonic functions, independently originated in the early seventies in the works of D. Aharonov and H.S. Shapiro, Ph. Davis, S. Richardson and M. Sakai. However, later it was discovered that the potential theoretic equivalent of the concept of a quadrature domain was clearly formulated already in a seminal memoir of Herglotz in 1914 (*Über die analytische Fortsetzung des Potentials ins Innere der anziehenden Massen*, Gekrönte Preisschr. der Jablonowskischen Gesellsch. zu Leipzig, 56 pp.) and was developed further in the works of Schmidt and Wavre. The modern time authors were motivated by quite a diverse spectrum of problems in the theory of univalent functions of a complex variable, approximation theory, fluid mechanics and potential theory on Riemann surfaces. This interdisciplinary trend has continued, and has even been amplified over the decades. Today, we can add to the ramifications of the theory of quadrature domains chapters of partial differential equations in the complex domain, variational problems for PDE and free boundaries, completely integrable systems related to quantum physics, gravitational lensing and many new aspects of the modern theory of fluid dynamics.

The theme of quadrature domains has been largely cultivated for more than three decades by Professor Shapiro and his PhD students. His 1992 book "*The Schwarz function and its generalization to higher dimensions*", Univ. Arkansas Lecture Notes Math. vol. 9, Wiley, New York, well illustrates the state of the field at the time of its writing. The present collection of articles reflects some of the progress that has been made in the last decade. However, the subject is so much alive nowadays that even in the short time that has passed between the occasion of the conference and the publication of this volume, new striking appearances of quadrature domains in mathematical physics have emerged.

The book contains both original articles and survey papers covering quite a wide scope of ideas in classical and modern analysis and applications. The survey articles written by the leading experts in the field will help to orient the beginners in the vastly increasing literature on the subject. The monograph may also help young researchers and graduate students wanting to familiarize themselves with this active and beautiful area of analysis that thrives on techniques from potential theory, complex analysis, geometry and partial differential equations. The book concludes with a selection of some open problems that were discussed in a special session at the conference. We hope that the book will attract young analysts looking for interesting and easily formulated questions, though still deep and important for numerous applications. On the other hand, the experts may find it helpful as a reference for the current status of the subject as well.

We were all inspired by Harold Shapiro's contagious enthusiasm, energy and never ending interest in finding new paths and beautiful problems in all areas of mathematics. We wholeheartedly wish him, on this occasion, a very happy birthday and many fruitful, healthy, and productive years ahead.

We are grateful to the National Science Foundation, University of Arkansas and Universities of California at Santa Barbara and San Diego for financial support of the conference.

The editors

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Open Problems Related to Quadrature Domains

1. Two questions concerning quadrature domains

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1.1. The a_3 problem

The a_2 problem was presented by Harold S. Shapiro in [4] and was finally solved in [2]. The similar "a_3 problem" (meaning replacing the condition $a_2(f) = \alpha,\ 1/2 < \alpha < 2$, by $a_3(f) = \alpha,\ 1/3 < \alpha < 3$) seems to be essentially harder. Indeed, not as for the "a_2 problem", here symmetrization does not seem to help and one needs to find other method to replace it.

Partial results. Any extremal f has a bounded derivative, namely $|f'| < M < \infty$ in the unit disk U, for some constant M. From this one easily deduces that all complex moments

$$\int_D w^n dA(w) = 0, \quad n \geq 3,$$

vanish, where D is the image domain $f(U)$ and dA stands for the area measure.

We note that uniqueness is not clear at all.

One would like to prove, like for the "a_2 problem" that $\tilde{D} = \mathrm{int}\overline{D}$ (i.e. the domain D with "erased slits") is a Jordan domain $\tilde{D}$.

If this could be done, then it would follow that there are absolute constants A, B, C with the property

$$\int_{\tilde{D}} g(w) dA(w) = Ag(0) + Bg'(0) + Cg''(0),$$

for any analytic, Lebesgue integrable function g in $\tilde{D}$. In other words, $\tilde{D}$ is a quadrature domain. From this point on, it should not be hard to find the complete solution.

1.2. The nature of the boundary of a generalized quadrature domain

It was shown in [1] that the boundary of a quadrature domain is algebraic. This was done by investigating the Schwarz function, based on a standard process of eliminating its poles. For generalized quadrature domains, the situation is more complicated.

To be more specific, we restrict ourselves to the case of a domain D arising in the recent article [3]:

$$\int_D g(w)dA(w) = Ag(0) + B\int_0^b g(u)du,$$

where g is an analytic, Lebesgue integrable function in D, the points $0, b$ belong to D and the line integral is taken along any curve inside D.

The case of a simply connected domain D was resolved in [3].

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2. Quadrature domains of infinite order and infinite connectivity

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The simplest example of a bounded quadrature domain is a circular disc of radius r. Such a domain D satisfies the quadrature identity

$$\int\int_D h(z)d\sigma = \pi r^2 h(z_0)$$

where z_0 is the centre of the disc, $h(z)$ is a suitable analytic function integrable over D and $d\sigma$ denotes area measure.

Any *finite* collection of disconnected circular discs also represents a quadrature domain. Connected quadrature domains, of finite order and finite connectivity, can be constructed by "continuing" such collections of disconnected discs (see, for example, [Crowdy, *Proc. Roy. Soc. A*, **457**, (2001)] where such ideas have been used in applications to vortex equilibria).

Consider $2N + 1$ identical circular discs of radius $1/2$ centred at $z = -N$, $-(N - 1), \ldots, -1, 0, 1, 2, \ldots, N$. The circular discs touch and the disconnected bounded domain D_N, say, satisfies the (order $2N + 1$) quadrature identity

$$\int\int_{D_N} h(z)d\sigma = \sum_{k=-N}^{N} \pi r^2 h(k)$$

with $r = 1/2$. For $r > 1/2$, the domain D_N "continues" to a simply-connected quadrature domain of order $2N + 1$. Conformal maps from a unit circle to such a domain are known to be rational functions.

It is natural to consider the following limit: let the number of circular discs centred at the integers tend to infinity, i.e., consider the domain of disconnected discs D_∞. Such a domain will be unbounded. Provided $h(z)$ decays sufficiently fast at infinity, it will also satisfy the "generalized" quadrature identity

$$\int\int_{D_\infty} h(z)d\sigma = \sum_{k=-\infty}^{\infty} \pi r^2 h(k),$$

which has *infinite* order. Yet, it can still be considered a quadrature domain in the sense that the set of quadrature data is *finite.*

The question arises if D_∞ can be similarly "continued" to a *connected* unbounded quadrature domain of infinite order when $r > 1/2$. This question can be answered by direct construction (see, Crowdy (unpublished notes)). A natural extended definition is to consider quadrature domains satisfying

$$\int\int_{D_\infty} h(z)d\sigma = \sum_{n=1}^{N} \sum_{k_n=-\infty}^{\infty} \pi r_n^2 h(z_n + k_n) \tag{1}$$

which, although of infinite order, depend on just a finite set of quadrature data $\{r_n, z_n \in \mathbb{C}\}$ where, by the expected 1-periodicity of the configuration, z_n can be taken in the interval $-1/2 < |z_n| < 1/2$.

In a similar way, it is possible to envisage infinite-order quadrature domains of infinite connectivity, but still with a finite set of quadrature data. This can be done by defining the bounded domain $D_{M,N}$ of identical touching disconnected discs in a square array centred on the points $z_n = m + in$ where m, n are integers $|m| \leq M, |n| \leq N$. Taking the limits $M \to \infty$, $N \to \infty$ and "continuing" to a connected domain produces $D_{\infty,\infty}$ — an unbounded quadrature domain of infinite order and infinite connectivity satisfying

$$\int\int_{D_{\infty,\infty}} h(z)d\sigma = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \pi r^2 h(m + n)$$

which will be (infinitely) connected for r just greater than $1/2$. Generalized quadrature identities analogous to (1) can be envisaged.

The following questions arise given these "generalized" infinite-order quadrature domains:

- Can it be proven rigorously that such domains exist?
- If they do, what it the best way to parametrize/uniformize the boundaries? What can be said about the analytic structure of conformal mappings to, for example, a simple "unit cell" in these singly/doubly-periodic infinite-order quadrature domains?
- Bounded, finite-order quadrature domains have algebraic boundaries. What can be said about the boundaries of the generalized domains?
- What can be said about the Cauchy transforms of the generalized domains?

These classes of domains are potentially of great use in applications.

3. Problems on quadrature domains

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3.1. Uniqueness of simply connected quadrature domains

For "classical" quadrature domains [1], [9], [22] in two dimensions the question may be stated as follows: Do there exist two different simply connected domains $\Omega \subset \mathbf{C}$ ($\Omega = \Omega_1, \Omega_2$ say) admitting one and the same quadrature identity

$$\iint_\Omega f(z)dxdy = \sum_{j=1}^{m} \sum_{k=0}^{n_j-1} c_{jk} f^{(k)}(z_j),$$

to hold for every analytic and integrable function f in Ω?

Variants of the question are obtained by replacing the right-hand side by $\int f d\mu$ for a measure μ with compact support and working with harmonic test functions instead of analytic ones. In higher dimension the assumption of simple connectivity should be replaced by the assumption that the domains are "solid", meaning that the complement of the closure of the domain in question has only one component (the unbounded one).

The above uniqueness question is essentially equivalent to the exterior inverse problem in potential theory: Does there exist two different solid domains such that their Newtonian potentials agree outside their union? (The domains are considered as bodies of density one.)

See e.g. [26], [16] for general information about this inverse problem and [10] for its relation to quadrature domain theory. See also the last section (on open problems) in [24].

If one allows suitable weights (in place of pure Lebesgue measure) there are counterexamples for the inverse problem of potential theory, see [18]. Similarly, if the potentials are asked to agree only in a neighbourhood of infinity [20], [26]. There are also counterexamples for the corresponding question when area measure (in the two dimensional case) is replaced by arclength measure on the boundary, [15], Proposition 6.2.

Therefore one might conjecture that there is no uniqueness in the above stated problems. Nevertheless it would be good to have a definite answer.

3.2. On algebraic boundaries

Given points $x_j \in \mathbf{R}^n$ and coefficients $c_j > 0$ ($j = 1, \ldots, m$ say) it is known [21], [10], [11] that there exists an open set (possibly disconnected) $\Omega \subset \mathbf{R}^n$ admitting the quadrature identity

$$\int_\Omega h dx = \sum_{j=1}^{m} c_j h(x_j)$$

for all integrable harmonic functions h in Ω. It is uniquely determined (up to nullsets) if the appropriate inequality ($\geq$) is required to hold for all subharmonic test functions.

It is known from general theory for free boundaries [7] that $\partial\Omega$ is a real analytic hypersurface, except possibly for a small set of singular points (see more precisely [2]). In two dimensions it is moreover known [1], [9], [22] that $\partial\Omega$ is algebraic. For example, if $r > 1$ in the two point quadrature identity (for $\Omega \subset \mathbf{C}$)

$$\int_\Omega h dx dy = \pi r^2 (h(-1) + h(1)),$$

then $\partial\Omega$ is given by

$$(x^2 + y^2)^2 - 2r^2(x^2 + y^2) - 2(x^2 - y^2) = 0.$$

See e.g. [22], Section 3.1, and for more examples [5], [4].

Now the question is whether anything similar is true in higher dimension? Is $\partial\Omega$ necessarily algebraic or belongs to some other special class of surfaces? Even for two point quadrature domains in higher dimensions no explicit description of the boundary is known.

There are known a few examples of quadrature domains in $\mathbf{R}^4$ which do have algebraic boundaries, see [17], [24].

3.3. Quadrature domains and Hele-Shaw flow with weights, or on Riemannian manifolds

Let $\rho > 0$ be a suitably smooth weight function in $\mathbf{R}^n$. Then for any $t > 0$ there exists a unique (up to nullsets) domain $\Omega_t \subset \mathbf{R}^n$ such that

$$th(0) \leq \int_{\Omega_t} h\rho dx$$

for every integrable subharmonic function h in Ω_t (if h is harmonic, the inequality will become equality).

The presence of the time parameter t allows for the interpretation that $\{\Omega_t : t > 0\}$ is the solution of a Hele-Shaw flow moving boundary problem with an injection point [19], [8], [11] and the weight ρ allows for the interpretation that everything takes place on a Riemannian manifold.

It was shown in [13] and [14] that if, in the case of two dimensions, the weight ρ is logarithmically subharmonic, i.e. $\Delta \log \rho \geq 0$, then all the Ω_t are simply connected. In the view of Riemannian manifolds the logarithmic subharmonicity means that the Gaussian curvature of the manifold is nonpositive (hyperbolic manifold). The result can be used to provide the hyperbolic manifold with global polar coordinates. The radii in these polar coordinates can be thought of as Hele-Shaw geodesics and with respect to these the fluid domains Ω_t are starlike. See [14] for further discussions.

It is natural, and probably rewarding, to search for corresponding results in higher dimensions.

A few more results on the geometry of Hele-Shaw flow on manifolds (this time surfaces in $\mathbf{R}^3$) can be found in [24], Section 7.

Thus we expect that there is a rich theory concerning the geometry of quadrature domains and Hele-Shaw flows on manifolds to be discovered.

3.4. Parabolic quadrature domains

Parabolic potential theory is already well-established (see e.g. [6]) and there are some examples and beginning of theory for quadrature domain type questions. See for example [23] and [12]. Probably a general theory can be developed.

3.5. Quadrature domain theory based on Einstein's theory of gravitation rather than Newton's theory ("hyperbolic quadrature domains")

This programme is very tentative and it may very well be that it makes no sense to look for quadrature domains in full space-time. However, restricting to space-like slices we are still within the range of (nonlinear) elliptic potential theory, and it is quite likely that quadrature domain theory could be useful for certain problems there, exactly as it has turned out to be useful in a variety of problems in fluid dynamics [3]. One example is the type of problem discussed in [25].

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4. A question on the regularity of the boundary of a quadrature domain

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This open problem deals with the regularity of the boundary of quadrature domains.

Let Ω be a bounded quadrature domain for a measure μ and for the class of harmonic integrable function in Ω. By this we mean that

$$U^{\Omega}(x) = U^{\mu}(x) \quad \text{and} \quad \nabla U^{\Omega}(x) = \nabla U^{\mu}(x) \quad \text{for } x \in \mathbf{R}^n \setminus \Omega,$$

where $\mathrm{supp}(\mu) \subset \overline{\Omega}$, U^{μ} denotes the Newtonian potential of μ and U^{Ω} is the Newtonian potential of the characteristic function of Ω.

Theorem 1. *Let Ω be a quadrature domain for a measure μ, $x_0 \in \partial\Omega$ and assume $\{x : |x - x_0| < r\} \cap \mathrm{supp}(\mu) = \emptyset$ for some positive r. If the complement of Ω, $\mathbf{R}^n \setminus \Omega$, is not "too thin" near x_0, then $\partial\Omega$ is a real analytic surface in a neighborhood of x_0.*

See Sakai [4], Gustafsson and Putinar [2] for $n = 2$, and Caffarelli, Karp and Shahgholian [1] for $n \geq 3$. The second result deals with the case where the support of the measure μ meets the boundary $\partial\Omega$ in a certain way (see Karp and Margulis [3]).

Theorem 2. *Let Ω be a quadrature domain for a measure μ, $0 \in \partial\Omega$ and assume*

$$\int_{\{|x|<r\}} \frac{d|\mu|(x)}{|x|^n} < \infty. \tag{2}$$

If there is a sequence $\{\varepsilon_k\} \searrow 0$ such that $\lim_k \chi_{\Omega}(\varepsilon_k x) = \chi_K(x)$, where K is a cone and $\mathrm{supp}(\mu) \subset K$, then the cone K equals to a half-space.

In other words, if the measure μ hits the boundary in a "soft" way (i.e. (2) holds), then the boundary $\partial\Omega$ has, in a certain sense, a tangent plane at $0 \in \partial\Omega$.

Open Problem: *Prove that under the assumptions of Theorem 2 the boundary is a real analytic surface in a neighborhood of $\{0\}$.*

The solution of that problem will show that either the measure μ hits the boundary of a quadrature domain strongly such that

$$\int_{\{|x|<r\}} \frac{d|\mu|(x)}{|x|^n} = \infty, \tag{3}$$

or it does not hit the boundary at all.

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5. Two questions in potential theory

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5.1. Bounded point evaluation for harmonic polynomials

Problem 1. *Suppose a measure μ supported in the unit ball B in $\mathbf{R}^n, n \geq 3$, annihilates harmonic polynomials. Does there exist a set E in B of positive volume such that for each point x in E one could find a measure $\nu_x, \nu_x \ll \mu$, which represents all harmonic polynomials at x, i.e.,*

$$u(x) = \int u d\nu_x$$

for all harmonic polynomials u.

As is shown in [KS1], the affirmative answer would immediately, by using the argument in [KS1], imply the uniqueness of the best uniform harmonic approximation to functions continuous on B, thus answering in the affirmative an old question of W. Hayman. In $\mathbf{R}^2$ the answer is "yes" and follows at once from the Cauchy transform techniques (cf. [KS1]). The difficulty in higher dimensions is that unlike analytic polynomials one cannot "divide" harmonic polynomials.

Along the same line of thought the following potential-theoretic question seems very attractive (cf. [KS1].)

Problem 2. *If a real measure μ (considered as a charge distribution on B) produces the same electric field outside B as does a positive point charge at some point w in B, then there is also a (positive) charge distribution m with $m \ll |\mu|$ producing the same field outside B. (The absolute continuity implies, in particular, that m is not permitted to place charges anywhere off supp μ.)*

This is again true in the plane [KS1], but the argument relies on Banach algebras techniques (harmonic polynomials are real parts of analytic polynomials there!) and does not extend to higher dimensions.

Yet, the result seems not unreasonable, on "physical" grounds also in $n > 2$ dimensions. Indeed, what we seem to be looking for is simply to ignore the negative charges of μ while at the same time redistributing the positive charges within the support of μ.

5.2. Global behaviour of the Schwarz potentials

Let Γ be a real analytic nonsingular surface in $\mathbf{R}^n$. Let U be the function defined locally near Γ, having its Laplacian equal to 1 and vanishing on Γ together with its first derivatives. (The existence of such U follows at once from the Cauchy–Kovalevskaya theorem.) This function U (cf. [KS2]), often called the *modified Schwarz potential* of Γ, is the basic tool in almost all questions associated with quadrature domains for harmonic functions(cf. [KS2]). It naturally extends to $\mathbf{R}^n$ the concept of the Schwarz function in the plane (cf. [Da]).

Conjecture. *If U is an entire real-analytic function in $\mathbf{R}^n$, then Γ must be a plane.*

The similar result due to Ph. Davis [Da] holds in the plane and is easy to prove. In $\mathbf{R}^n$ it was proved in [KS2] for the case when U is a polynomial. Later on Karp and Shahgholian extended the argument to a slightly more general situation, but the main bulk of the conjecture remains open. Interestingly, G. Johnsson constructed simple examples in $\mathbf{C}^n$ of analytic hypersurfaces for which the corresponding modified Schwarz potentials are entire, or even polynomials (cf. [K]). The feeling, why the Conjecture may still hold in $\mathbf{R}^n$ relies on a general (non-proven) consent that the singularities of U in $\mathbf{R}^n$ reflect the appearance of nonzero curvature on Γ. However, even in two dimensions, no proof of Davis' theorem based on calculating curvature via the Schwarz potential is known.

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6. Linear data of a quadrature domain

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Let $\Omega \subset \mathbf{C}$ be a bounded quadrature domain with d nodes, counting multiplicities. Let $p(z)$ be the monic polynomial of degree d vanishing at these quadrature nodes. Then it is known [GP] that the defining equation of $\partial\Omega$ has the form

$$P(z, \overline{z}) = |p(z)|^2 - \sum_{k=0}^{d-1} |p_k(z)|^2,$$

where $p_k(z)$ are polynomials of degree k. Moreover, simple linear algebra arguments show that there exists a $d \times d$ matrix A, with a cyclic vector ξ, such that

$$P(z, \overline{z}) = |p(z)|^2 (1 - \|(A - z)^{-1}\xi\|^2). \tag{4}$$

In this case, the spectrum of A coincides with the zeros of p and the quadrature nodes of Ω. In this way, a remarkable interplay between the properties of A and those of Ω can be established.

An indirect characterization of the pairs (A, ξ) of square matrices with a distinguished cyclic vector corresponding to a quadrature domain is available: Let A be the restriction of a pure co-hyponormal operator T with rank-one self commutator $[T, T^*] = \xi \otimes \xi$ to the finite dimensional subspace $\vee_{j=0}^{\infty} T^j \xi$. Moreover, a block-diagonal matricial representation of T is then derivable by simple operations from (A, ξ), see [P1, P2]. This is similar to a Jacobi matrix decomposition of T.

However, an intrinsic characterization of the pairs (A, ξ) (attached to quadrature domains) among the variety of all matrices with a cyclic vector is not known. Examples of 2×2 such matrices can be found in [GP].

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7. Fischer's problem

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Let **F** be the set of holomorphic functions on $\mathbf{C}^n$ such that

$$\int_{\mathbf{C}^n} |f|^2 e^{-|z|^2} d\lambda < +\infty$$

where $d\lambda$ is the Lebesgue measure on $\mathbf{C}^n$, and for $z = (z_1, \dots, z_n) \in \mathbf{C}^n$, $|z|^2 = |z_1|^2 + \cdots + |z_n|^2$. We denote by $\mathrm{H}(\mathbf{C}^n)$ the space of holomorphic functions defined on $\mathbf{C}^n$, by $\mathrm{Exp}(\mathbf{C}^n)$ the space of holomorphic functions on $\mathbf{C}^n$ of exponential type which means that

$$\exists C > 0, \qquad \forall z \in \mathbf{C}^n, \qquad |f(z)| \leq Ce^{C|z|}.$$

Let P and Q be polynomials in $\mathbf{C}[z_1, \dots, z_n]$. If P is of the form $\sum a_\alpha z^\alpha$ with the usual multi-index notation, we denote by $P(D)$ the holomorphic differential operator with constant coefficients $\sum a_\alpha \frac{\partial}{\partial z^\alpha}$.

A will denote one of the three functions spaces defined as above, which means that $\mathbf{A} = \mathbf{F}$ or $\mathbf{A} = \mathrm{H}(\mathbf{C}^n)$ or $\mathbf{A} = \mathrm{Exp}(\mathbf{C}^n)$.

The problem is the following: *given g and h in **A** find the necessary and sufficient conditions on P and Q in order that there is a unique $f \in \mathbf{A}$ such that*

$$\begin{cases} P(D)f = g \\ Q|(f-h). \end{cases}$$

The notation $\phi|\psi$ in **A** means that there exists $\theta \in \mathbf{A}$ such that $\psi = \phi\theta$. Moreover, we want to find an explicit expression of the solution f in terms of residue currents, g and h (*cf.* [Pa], [Ri]). This kind of problem is called by Hörmander a "Mixed Cauchy Problem" ([Eb-Sh1], [Eb-Sh2]). An obvious necessary condition is $\deg P = \deg Q$ ([Me-Yg]).

If we are interested in the case $Q = z_1^m$, the condition $Q|(f-h)$ means that the derivatives of f of order less or equal to $m-1$ are the the derivatives of h on the hyperplane $\{z_1 = 0\}$, and we are in the situation of the usual Cauchy Problem. In this case, we have explicit formulas.

If we are now interested in the case $Q = z_1^{m_1} z_2^{m_2}$, then the condition $Q|(f-h)$ means that the derivatives of f of order less or equal to $m_1 - 1$ with respect to z_1 and the derivatives of f of order less or equal to $m_2 - 1$ with respect to z_2 are the derivatives of h on the hyperplanes $\{z_1 = 0\}$ and $\{z_2 = 0\}$. We see in this case that we are not in the situation of the usual Cauchy Problem.

If $\mathbf{A} = \mathbf{F}$ and if $Q = P^*$, which means that $Q(z) = \overline{P(\bar{z})}$ then it is not too hard to see that the mixed Cauchy Problem above as a unique solution ([Me-Yg]). To see the history of Fischer's problem, more references and applications, one can also consult [An].

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8. Two problems on quadrature domains

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8.1. Estimates of the number of special points

This is a problem proposed by Harold S. Shapiro at the University of Maryland in 1985 when we were discussing quadrature domains, see [5].

Let Ω be a bounded domain in the complex plane surrounded by a finite number of nondegenerate curves and let $z_1, \dots, z_m$ be distinct points in Ω. Assume that

$$\iint_\Omega f(z)dxdy = \sum_{j=1}^{m} \sum_{k=0}^{n_j-1} c_{jk} f^{(k)}(z_j) \qquad (z = x + iy)$$

holds for every holomorphic and integrable function f in Ω. Then the boundary $\partial\Omega$ of Ω is algebraic and has possibly a finite number of cusps and double points on it, see [1]. We assume that $c_{jn_j-1} \neq 0$ for every j. The Schwarz function S for $\partial\Omega$ has $n = \sum_{j=1}^{m} n_j$ poles in Ω. We are interested in points z in Ω satisfying

$$S(z) = \overline{z}$$

called "special points" by Shapiro. They play important roles in studying the quadrature domain Ω.

Let s be the number of special points, a the connectivity of Ω, c the number of cusps on $\partial\Omega$ and d the number of double points on $\partial\Omega$. The problem is to find the relations between n, s, a, c and d.

There have been several results since then:

$$s \leq (n-1)^2 + 1 - a - c - 2d$$

by Gustafsson [2], see also McCarthy and Yang [3], and

$$s \geq n - 2 + a - c$$

by myself [4], but it seems that we have not yet obtained good estimates of s.

8.2. Quadrature domains for holomorphic functions of several variables

Discuss quadrature domains for holomorphic functions of several complex variables and establish a theory of such domains.

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9. Generalized quadrature domains

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In this section I will present some new directions in the theory of quadrature domains (QD). Two possible ways of extending this notion will be discussed in more detail: *parabolic QD* and *QD with partially fixed boundary.* Both of these problems are related to some of my recent research.

9.1. The original problem

A quadrature domain is a domain $\Omega \in \mathbf{R}^n$ with the property that for a given measure μ (with support in $\overline{\Omega}$) and an appropriately defined class of functions $\mathcal{A}$ one has the integral identity:

$$\int_\Omega h(x)dx = \int h(x)d\mu \qquad \forall h \in \mathcal{A}.$$

The deep connection between QD and variational problems (and or complementary problems), discovered by M. Sakai, has made it possible to study such problems from a variational inequality point of view. Although, not all QD can be studied within the framework of variational analysis it is still one of the main tools for deep analysis of such problems.

My objective here is to present two (these are partially new) directions of study in free boundary problems and give a QD-formulation of them. I hope that in this way one can find some new tools in solving certain problems that I will pose below.

9.2. Parabolic QD

In this part I will discuss a generalized form of QD. So let f, and μ, be non-negative functions (measures), then find a bounded $\Omega \subset \mathbf{R}^{n+1}$ with $\mathrm{supp}(\mu) \subset \overline{\Omega}$ and such that

$$\int\int_\Omega f(x,t)h(x,t)\ dxdt \ = \int\int h(x,t)\mu(x,t)\ dxdt\ , \tag{5}$$

for all integrable caloric functions h in Ω. The particular case of μ being the Dirac measure at (x^0, t^0) gives the mean-value property for caloric functions with respect to the point (x^0, t^0). One also observes immediately that, by (9.1) $\Omega \subset \mathbf{R}^n \times (-\infty, t^0)$.

A, probably, well-known identity of the type (9.1) with $f = |x|^2/t^2$ and μ a multiple of the Dirac mass is given by the so-called heat balls, which are the level sets of the fundamental solution to the adjoint heat operator $\Delta + D_t$. For a nice presentation of the mean value property for heat balls see the reader-friendly book by L.C. Evans [E].

9.3. Reformulation

Reflect the domain Ω in $\{t = 0\}$ so that the identity will now hold for anti-caloric functions. Therefore we assume that

$$\int\int_\Omega h(x,t)f(x,t)\ dxdt \ = \int\int h(x,t)\mu(x,t)\ dxdt \tag{6}$$

for all integrable anti-caloric functions h in Ω; here we used the same notation Ω for the reflected domain.

A couple of assumptions are in order now:

- $\Omega \subset \mathbf{R}^n \times (0, \infty)$.

- Set $\mu = g + g_0$ where
$$\operatorname{supp}(g) \subset \overline{\Omega} \cap \{t > 0\} \quad \text{and} \quad \operatorname{supp}(g_0) \subset \{t = 0\}.$$

In this way we separate the initial data g_0 and the heat source g.

Set
$$u(x,t) = \int\int_\Omega K(x-y,t-s)\mu(y,s)dyds - \int\int K(x-y,t-s)f(y,s)dyds,$$
where
$$K(z,\tau) = (4\pi\tau)^{-n/2}\exp\left(-|z|^2/(4\tau)\right), \qquad \tau > 0,$$
and $K(z,\tau) = 0$ for $\tau \leq 0$.

Now one readily verifies using identity (9.2) that
$$\begin{cases} \Delta u - D_t u = f\chi_\Omega - g & \text{in } \mathbf{R}_+^{n+1}, \\ u = 0 & \text{in } \mathbf{R}_+^{n+1} \setminus \Omega, \\ u(x,0) = g_0, & \\ \operatorname{supp} g \subset \overline{\Omega}, & \end{cases} \tag{7}$$
where
$$\mathbf{R}_+^{n+1} = \mathbf{R}^n \times (0,\infty),$$
and the differential equation is interpreted in the weak or distributional sense.

Hence we have a connection between parabolic-QD and the free boundary formulation above. For some details see [ASU]. The above setting appears in several problems in mathematical physics, e.g., in the Stefan problem.

9.4. QD with partially fixed boundary

Another way of generalizing the notion of QD, is to considering part of the boundary fixed. The easiest way is to start from the free boundary formulation and go back to QD.

For simplicity we work in the upper half space $\mathbf{R}_+^n = \{x_n > 0\}$, where the plane $\{x_n = 0\}$ will be considered as the fixed part of the boundary.

As before for a given measure μ with support in $\mathbf{R}_+^n$, find a function u and a domain $\Omega \subset \mathbf{R}_+^n$ (usually the only way to find the domain is to have $\Omega = \{u > 0\}$) such that
$$\Delta u = \chi_\Omega - \mu \qquad \text{in } \mathbf{R}_+^n$$
and $u = \nabla u = 0$ in $\Omega^c \cap \mathbf{R}_+^n$. Now on the boundary $\{x_n = 0\}$ one needs to assign values for u
$$u(x',0) = f(x'). \tag{8}$$
We simplify the problem even further by assuming $f \equiv 0$. Now one can reformulate the above problem by Green's identity as
$$\int_\Omega h dx = \int h d\mu$$
for all integrable, harmonic functions h (in Ω), vanishing on $x_n = 0$.

9.5. Open questions

Let us now present some open questions around the topic of QD. I will consider problems related to the standard QD and also the above variants.

Problem 1 (*Null QD/ Global solutions*)
A null quadrature domain Ω is defined through the identity

$$\int_\Omega h dx = 0$$

for all integrable harmonic functions in Ω. Reformulation of this in terms of global solutions to the pde

$$u(\Delta u - 1) = 0, \qquad |u(x)| \leq C|x|^2.$$

For details see [Sh] and [CKS]. It was conjectured in [Sh] that global solutions are limit domains of the exterior of ellipsoids. The corresponding problem in two space dimensions was settled by M. Sakai. Later H.S. Shapiro gave another proof. Global solutions in half-spaces were completely classified in [SU]). They appear to be one dimensional.

So we repose the question whether one can prove the conjecture in [Sh]. It is noteworthy that by classifications of [CKS] (and earlier results on QD with bounded complements), it only remains to show that Null QD which are thin at infinity must be paraboloids.

Problem 2 (*Parabolic QD.*)
For parabolic QD it was recently shown that these solutions are convex (see [CPS]). There are many questions still open around the parabolic QD, and the subject has just been born!

- Existence question,
- Regularity issues (near the initial datum),
- Complete classification of global solutions/null QD, for the parabolic case,
- Asymptotic behavior of solutions, as time grows: finite time extinction.

Problem 3 (*QD with partially fixed boundary.*)
There are many problems related to QD with fixed boundaries. Most of these results concern regularity issues, the behavior of solutions and the free boundary. I refer to [SU], for the elliptic case and to [ASU] for parabolic case. (These papers treat the case $f \equiv 0$.)

Question of interest are the behavior of the solution/free boundary when f in (8) is nonzero. E.g., when f is $C^{1,1}$ one can relate the problem to that of the Dam problem of water reservoirs.

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10. Further study of quadrature domains on curved surfaces

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In the recent work of H. Hedenmalm and S. Shimorin, they study mean value disks on hyperbolic Riemann surfaces. These can be interpreted as domains of the Hele-Shaw flow starting at a point and at the same time they are the simplest examples of quadrature domains on curved surfaces. Even for these domains, some natural questions are still open, for example, it is not known if they are geodesically star-shaped. Another interesting question is how far the assumption on hyperbolicity of the metric can be relaxed so that mean value disks still remain topologically trivial. Extensions of the results to dimensions higher than 2 is also an open problem.

A further possible direction of research is the study of mean value circles on curved surfaces. This corresponds to the following problem in the plane: given a weight function $\omega(z)$ defined in the plane, one searches for closed simple curves γ_t having the property

$$\int_{\gamma_t} h\omega \, ds = th(0)$$

for all bounded harmonic functions h.