

Jean-Pierre Serre

**Lectures on the
Mordell-Weil Theorem**

Aspects Of Mathematics

Edited by Klas Diederich

- Vol. E 3: G. Hector/U. Hirsch: Introduction to the Geometry of Foliations, Part B
- Vol. E 5: P. Stiller: Automorphic Forms and the Picard Number of an Elliptic Surface
- Vol. E 6: G. Faltings/G. Wüstholz et al.: Rational Points*
- Vol. E 9: A. Howard/P.-M. Wong (Eds.): Contribution to Several Complex Variables
- Vol. E 10: A. J. Tromba (Ed.): Seminar of New Results in Nonlinear Partial Differential Equations*
- Vol. E 15: J.-P. Serre: Lectures on the Mordell-Weil Theorem
- Vol. E 16: K. Iwasaki/H. Kimura/S. Shimomura/M. Yoshida: From Gauss to Painlevé
- Vol. E 17: K. Diederich (Ed.): Complex Analysis
- Vol. E 18: W. W. J. Hulsbergen: Conjectures in Arithmetic Algebraic Geometry
- Vol. E 19: R. Racke: Lectures on Nonlinear Evolution Equations
- Vol. E 20: F. Hirzebruch, Th. Berger, R. Jung: Manifolds and Modular Forms*
- Vol. E 21: H. Fujimoto: Value Distribution Theory of the Gauss Map of Minimal Surfaces in \mathbf{R}^m
- Vol. E 22: D. V. Anosov/A. A. Bolibruch: The Riemann-Hilbert Problem
- Vol. E 23: A. P. Fordy/J. C. Wood (Eds.): Harmonic Maps and Integrable Systems
- Vol. E 24: D. S. Alexander: A History of Complex Dynamics
- Vol. E 25: A. Tikhomirov/A. Tyurin (Eds.): Algebraic Geometry and its Applications
- Vol. E 26: H. Skoda/J.-M. Trépreau (Eds.): Contributions to Complex Analysis and Analytic Geometry
- Vol. E 27: D. N. Akhiezer: Lie Group Actions in Complex Analysis
- Vol. E 28: R. Gérard, H. Tahara: Singular Nonlinear Partial Differential Equations
- Vol. E 29: R.-P. Holzapfel: Ball and Surface Arithmetics
- Vol. E 30: R. Huber: Étale Cohomology of Rigid Analytic Varieties and Adic Spaces

Jean-Pierre Serre

Lectures on the Mordell-Weil Theorem

Translated and edited by Martin Brown
from notes by Michel Waldschmidt

3rd edition

Springer Fachmedien Wiesbaden GmbH



Prof. *Jean-Pierre Serre*
Collège de France
Chaire d'Algèbre et Géométrie
75005 Paris

AMS Subject Classification: 14 G 13, 14 K 10, 14 K 15

1st edition 1989
2nd edition 1990
3rd edition 1997

All rights reserved

© Springer Fachmedien Wiesbaden 1997

Originally published by Friedr. Vieweg & Sohn Verlagsgesellschaft mbH, Braunschweig/Wiesbaden, 1997



No part of this publication may be reproduced, stored in a retrieval system or transmitted, mechanical, photocopying or otherwise, without prior permission of the copyright holder.

Cover design: Wolfgang Nieger, Wiesbaden

Printed on acid-free paper

ISSN 0179-2156

ISBN 978-3-663-10634-0

ISBN 978-3-663-10632-6 (eBook)

DOI 10.1007/978-3-663-10632-6

Foreword

This is a translation of “Autour du théorème de Mordell-Weil”, a course given by J.-P. Serre at the Collège de France in 1980 and 1981.

These notes were originally written weekly by Michel Waldschmidt and have been reproduced by Publications Mathématiques de l’Université de Paris VI, by photocopying the handwritten manuscript.

The present translation follows roughly the French text, with many modifications and rearrangements. We have not tried to give a detailed account of the new results due to Faltings, Raynaud, Gross-Zagier ...; we have just mentioned them in notes at the appropriate places, and given bibliographical references.

Paris, Fall 1988

M. L. Brown
J.-P. Serre

 CONTENTS

1. Summary.	1
1.1. Heights.	3
1.2. The Mordell-Weil theorem and Mordell's conjecture.	3
1.3. Integral points on algebraic curves. Siegel's theorem.	4
1.4. Baker's method.	5
1.5. Hilbert's irreducibility theorem. Sieves.	5
2. Heights.	7
2.1. The product formula.	7
2.2. Heights on $\mathbf{P}_m(K)$.	10
2.3. Properties of heights.	13
2.4. Northcott's finiteness theorem.	16
2.5. Quantitative form of Northcott's theorem.	17
2.6. Height associated to a morphism $\phi : X \rightarrow \mathbf{P}_n$.	19
2.7. The group $\text{Pic}(X)$.	20
2.8. Heights and line bundles.	22
2.9. $h_c = O(1) \Leftrightarrow c$ is of finite order (number fields).	24
2.10. Positivity of the height.	24
2.11. Divisors algebraically equivalent to zero.	25
2.12. Example-exercise: projective plane blown up at a point.	26
3. Normalised heights.	29
3.1. Néron-Tate normalisation.	29
3.2. Abelian varieties.	31
3.3. Quadraticity of \tilde{h}_c on abelian varieties.	35
3.4. Duality and Poincaré divisors.	36
3.5. Example: elliptic curves.	39
3.6. Exercises on elliptic curves.	40
3.7. Applications to properties of heights.	41
3.8. Non-degeneracy.	42
3.9. Structure of $A(K)$: a preliminary result.	43
3.10. Back to §2.11 (c algebraically equivalent to zero).	44
3.11. Back to §2.9 (torsion c).	46

4. The Mordell-Weil theorem.	49
4.1. Hermite's finiteness theorem.	49
4.2. The Chevalley-Weil theorem.	50
4.3. The Mordell-Weil theorem.	51
4.4. The classical descent.	53
4.5. The number of points of bounded height on an abelian variety.	53
4.6. Explicit form of the weak Mordell-Weil theorem.	55
5. Mordell's conjecture.	58
5.1. Chabauty's theorem.	58
5.2. The Manin-Demjanenko theorem.	62
5.3. First application: Fermat quartics (Demjanenko).	66
5.4. Second application: modular curves $X_0(p^n)$ (Manin).	67
5.5. The generalised Mordell conjecture.	73
5.6. Mumford's theorem; preliminaries.	74
5.7. Application to heights: Mumford's inequality.	77
6. Local calculation of normalised heights.	81
6.1. Bounded sets.	81
6.2. Local heights.	83
6.3. Néron's theorem.	87
6.4. Relation with global heights.	89
6.5. Elliptic curves.	90
7. Siegel's method.	94
7.1. Quasi-integral sets.	94
7.2. Approximation of real numbers.	95
7.3. The approximation theorem on abelian varieties.	98
7.4. Application to curves of genus ≥ 1 .	101
7.5. Proof of Siegel's theorem.	102
7.6. Application to $P(f(n))$.	105
7.7. Effectivity.	106
8. Baker's method.	108
8.1. Reduction theorems.	108
8.2. Lower bounds for $\sum \beta_i \log \alpha_i$.	110
8.3. Application to $\mathbf{P}_1 - \{0, 1, \infty\}$.	112
8.4. Applications to other curves.	114
8.5. Applications to elliptic curves with good reduction outside a given finite set of places.	118

9. Hilbert's irreducibility theorem.	121
9.1. Thin sets.	121
9.2. Specialisation of Galois groups.	122
9.3. Examples of degrees 2,3,4,5.	123
9.4. Further properties of thin sets.	127
9.5. Hilbertian fields.	129
9.6. The irreducibility theorem: elementary proof.	130
9.7. Thin sets in \mathbf{P}_1 : upper bounds.	132
10. Construction of Galois extensions.	137
10.1. The method.	137
10.2. Extensions with Galois group S_n .	138
10.3. Extensions with Galois group A_n .	144
10.4. Further examples of Galois groups: use of elliptic curves.	145
10.5. Noether's method.	147
10.6. Infinite Galois extensions.	147
10.7. Recent results.	149
11. Construction of elliptic curves of large rank.	152
11.1. Néron's specialisation theorem.	152
11.2. Elliptic curves of rank ≥ 9 over \mathbf{Q} .	154
11.3. Elliptic curves of rank ≥ 10 over \mathbf{Q} .	158
11.4. Elliptic curves of rank ≥ 11 over \mathbf{Q} .	161
12. The large sieve.	163
12.1. Statement of the main theorem.	163
12.2. A lemma on finite groups.	164
12.3. The Davenport-Halberstam theorem.	166
12.4. Proof of the Davenport-Halberstam theorem.	167
12.5. End of the proof of the main theorem.	172
13. Applications of the large sieve to thin sets.	177
13.1. Statements of results.	177
13.2. Proof of theorem 1.	179
13.3. Proof of theorem 5.	183
13.4. Proof of theorem 3 from theorem 1.	186

Appendix: The class number 1 problem and integral points on modular curves.	188
A.1. Historical remarks.	188
A.2. Equivalent conditions for $h(-p) = 1$.	190
A.3. Orders of R_d .	191
A.4. Elliptic curves with complex multiplication.	192
A.5. Modular curves associated to normalisers of Cartan subgroups and their CM integral points.	194
A.6. Examples.	196
A.7. The Gel'fond-Linnik-Baker method.	197
Bibliography.	200
Index.	210