

# Applied Mathematical Sciences

Volume 171

## Series Editors

- A. Bloch, Department of Mathematics, University of Michigan, Ann Arbor, MI, USA  
C. L. Epstein, Department of Mathematics, University of Pennsylvania, Philadelphia, PA, USA  
A. Goriely, Department of Mathematics, University of Oxford, Oxford, UK  
L. Greengard, New York University, New York, NY, USA  
P. J. Holmes, Department of Mechanical and Aerospace Engineering, Princeton University, Princeton, NJ, USA

## Advisory Editors

- J. Bell, Center for Computational Sciences and Engineering, Lawrence Berkeley National Laboratory, Berkeley, CA, USA  
P. Constantin, Department of Mathematics, Princeton University, Princeton, NJ, USA  
R. Durrett, Department of Mathematics, Duke University, Durham, CA, USA  
R. Kohn, Courant Institute of Mathematical Sciences, New York University, New York, NY, USA  
R. Pego, Department of Mathematical Sciences, Carnegie Mellon University, Pittsburgh, PA, USA  
L. Ryzhik, Department of Mathematics, Stanford University, Stanford, CA, USA  
A. Singer, Department of Mathematics, Princeton University, Princeton, NJ, USA  
A. Stevens, Department of Applied Mathematics, University of Münster, Münster, Germany  
S. Wright, Computer Sciences Department, University of Wisconsin, Madison, WI, USA

## Founding Editors

- F. John, New York University, New York, NY, USA  
J.P. LaSalle, Brown University, Providence, RI, USA  
L. Sirovich, Brown University, Providence, RI, USA

The mathematization of all sciences, the fading of traditional scientific boundaries, the impact of computer technology, the growing importance of computer modeling and the necessity of scientific planning all create the need both in education and research for books that are introductory to and abreast of these developments. The purpose of this series is to provide such books, suitable for the user of mathematics, the mathematician interested in applications, and the student scientist. In particular, this series will provide an outlet for topics of immediate interest because of the novelty of its treatment of an application or of mathematics being applied or lying close to applications. These books should be accessible to readers versed in mathematics or science and engineering, and will feature a lively tutorial style, a focus on topics of current interest, and present clear exposition of broad appeal. A compliment to the Applied Mathematical Sciences series is the Texts in Applied Mathematics series, which publishes textbooks suitable for advanced undergraduate and beginning graduate courses.


More information about this series at <http://www.springer.com/series/34>

Laurent Younes

# Shapes and Diffeomorphisms

Second Edition

 Springer

Laurent Younes   
Center for Imaging Science  
Johns Hopkins University  
Baltimore, MD, USA

ISSN 0066-5452 ISSN 2196-968X (electronic)  
Applied Mathematical Sciences  
ISBN 978-3-662-58495-8 ISBN 978-3-662-58496-5 (eBook)  
<https://doi.org/10.1007/978-3-662-58496-5>

Library of Congress Control Number: 2018964233

Mathematics Subject Classification (2010): 68T10, 53-01, 68-02, 92C55, 37E30, 37C10, 53A04, 53A05, 68U05

1st edition: © Springer-Verlag Berlin Heidelberg 2010

2nd edition: © Springer-Verlag GmbH Germany, part of Springer Nature 2019

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer-Verlag GmbH, DE part of Springer Nature

The registered company address is: Heidelberger Platz 3, 14197 Berlin, Germany

*To Geneviève, Hannah, Salomé, Simon*

# Introduction to the Second Edition

Besides correcting some of the typos and mistakes from the first edition and implementing a few changes to the notation to make it uniform, several significant additions appear in the second edition. Most notable is the introduction of a discussion of optimal control theory in an infinite-dimensional framework (Appendix D), which is then used in multiple places to enrich the presentation of diffeomorphic matching, and a new chapter on shape datasets (Chap. 13 of the second edition). A few other changes have been made. These are listed below, where chapters are referred to as  $x/y$ , the first and second numbers indicating the first and second editions, respectively.

- Chapter 1/1: The presentation of closed curves and integrals along them has been revised. Some sections have been reordered to improve readability.
- Chapter 2/2: No significant changes.
- Chapter 3 of the first edition has been removed.
- Chapter 4/3: No major changes were made. The material on discrete surfaces was moved to Chap. 5/4, in which a new discussion is added on consistent discrete-to-continuous approximation.
- Chapters 6/5 and 7/6 only had minor modifications.
- Chapter 8/7: A few additional results on compositions of diffeomorphisms and their derivatives have been included. Small modifications were made to the rest of the chapter, which also relies more directly on results from the appendix.
- Chapter 9/8: The discussion on invariant operators and kernels has been updated and clarified.
- Chapter 10/9: Only minor changes were made in most sections. Exceptions are (1) curve and surface matching, in which a discussion of varifold distances was added, and (2) frame matching, which has been rewritten with a slightly different action, and a more rigorous handling of the supports of the frame fields.
- Chapter 11/10: Some of the discussion now directly uses results from the appendix on ODEs and optimal control. The proof of the existence and uniqueness of solutions of the EPDiff equation has also been rewritten.
- Chapter 12/11 had only minor changes.

- Chapter 13/12 on metamorphosis has been extended, with more examples, including in particular a detailed discussion of metamorphosis applied to the tangent representation of curves.
- Chapter 13 in the second edition is new.
- Only minor changes were made in Appendix A.
- Appendix B was extended with a discussion of differential forms and some consequences of Stokes's theorem.
- Appendix C has essentially the same content, even though part of the write-up was revised.
- Appendix D now includes an introduction to optimal control theory.
- Appendices E and F were only slightly modified.

# Introduction to the First Edition

Shape is a fascinating object of study. Understanding how a single shape can incur a complex range of transformations, while defining the same perceptually obvious figure, entails a rich and enticing collection of problems at the interface between applied mathematics, statistics, and computer science. Various applications in computer vision, object recognition, and medical imaging bring additional motivation for researchers to develop adequate theoretical background and methodology for solving these problems.

This book is an attempt at providing a description of the large range of methods that have been invented to represent, detect, or compare shapes (or more generally, deformable objects), together with the necessary mathematical background that they require. While certainly being a book on applied mathematics, it is also written in a way that will be of interest to an engineering- or computer-science-oriented reader, including in several places concrete algorithms and applicable methods, including experimental illustrations.

This book starts with a discussion of shape representation methods (Chapters 1–4), including classical aspects of the differential geometry of curves and surfaces, but borrowing also from other fields that have positively impacted the analysis of shape in practical applications, such as medial axes and discrete differential geometry.

The second part (Chapters 5–7) studies curve and surface evolution algorithms and how they relate to segmentation methods that can be used to extract shapes from images, using active contours or deformable templates. A reader with enough background in differential geometry may start reading this book at Chapter 6 or at Chapter 7 if the main focus of interest is on diffeomorphic registration and comparison methods.

In Chapters 7 and 8, basic concepts related to diffeomorphisms are introduced, discussing in particular how using ordinary differential equations associated with vector fields belonging to reproducing kernel Hilbert space provides a computationally convenient framework to handle them. Chapters 9 and 10 then focus on the registration of deformable objects using diffeomorphisms; in Chapter 9, we catalog a large collection of deformable objects and discuss matching functionals that can be used to compare them. Chapter 10 addresses diffeomorphic matching and



focuses in particular on methods that optimize a matching functional combined with a regularization term that penalizes the distance of a diffeomorphism to the identity within the group.

The next two chapters (11 and 12) discuss metric aspects of shape analysis, with a special focus on the relation between distances and group actions. Both the global and infinitesimal points of view are presented. The classical Kendall's metric over configurations of labeled points is included, as well as a short discussion of Riemannian metrics on plane curves. Chapter 12 provides a presentation of the theory of metamorphosis. Chapter 13 provides an introduction to the statistical analysis of shape data.

In the appendices are provided fundamental concepts that are needed in order to understand the rest of this book. The main items are some elements of functional analysis (Appendix A), of differential and Riemannian geometry (Appendix B), and of ordinary differential equations (Appendix C). Appendix D provides an introduction to optimization and optimal control. Appendix E focuses on principal component analysis and Appendix F on dynamic programming. In all cases, the appendices do not provide a comprehensive presentation of these theories, but simply what is needed in the particular context of this book.

Chapters 1 to 5, which are (with a few exceptions) rather elementary, provide an introduction to applied differential geometry that is suitable for an advanced undergraduate class. They can be combined with Chapter 6 to form a graduate-level class on the same subject. The first six chapters are written (with a few exceptions) in order to be accessible without using the more advanced features developed in the appendices. Chapters 8 to 13 represent specialized, advanced graduate topics.

I would like to thank my students and collaborators, who have helped to make the ideas that are developed in these notes reach their current state of maturation. I would like, in particular, to express my gratitude to Alain Trouvé and Michael Miller, whose collaboration over the last decade has been invaluable. Special thanks also to Darryl Holm, David Mumford, and Peter Michor. This book was written while the author was partially supported by the National Science Foundation, the National Institute of Health, and the Office for Naval Research.

# Contents

<b>1</b>	<b>Parametrized Plane Curves</b> . . . . .	<b>1</b>
1.1	Definitions . . . . .	1
1.2	Reparametrization Equivalence . . . . .	3
1.2.1	Open Curves . . . . .	3
1.2.2	Closed Curves . . . . .	4
1.3	Unit Tangent and Normal . . . . .	5
1.4	Embedded Curves . . . . .	6
1.5	The Integral Along a Curve and Arc Length . . . . .	7
1.6	Curvature . . . . .	11
1.7	Expression in Coordinates . . . . .	12
1.7.1	Cartesian Coordinates . . . . .	12
1.7.2	Polar Coordinates . . . . .	13
1.8	Euclidean Invariance . . . . .	14
1.9	The Frenet Frame . . . . .	15
1.10	Enclosed Area and the Green (Stokes) Formula . . . . .	17
1.11	The Rotation Index and Winding Number . . . . .	18
1.12	More on Curvature . . . . .	19
1.13	Discrete Curves and Curvature . . . . .	20
1.13.1	Least-Squares Approximation . . . . .	20
1.13.2	Curvature and Distance Maps . . . . .	23
1.14	Implicit Representation . . . . .	28
1.14.1	Introduction . . . . .	28
1.14.2	Example: Implicit Polynomials . . . . .	30
1.15	Invariance for Affine and Projective Transformations . . . . .	31
1.15.1	Euclidean Invariance . . . . .	32
1.15.2	Scale Invariance . . . . .	33
1.15.3	Special Affine Transformations . . . . .	35
1.15.4	Generalized Curvature . . . . .	36
1.15.5	Affine Arc Length . . . . .	40

1.15.6	Projective Arc Length . . . . .	42
1.15.7	Affine Curvature . . . . .	44
1.15.8	Projective Curvature . . . . .	45
1.16	Non-local Representations . . . . .	49
1.16.1	Semi-local Invariants . . . . .	49
1.16.2	The Shape Context . . . . .	50
1.16.3	Conformal Welding . . . . .	51
<b>2</b>	<b>The Medial Axis . . . . .</b>	<b>57</b>
2.1	Introduction . . . . .	57
2.2	The Structure of the Medial Axis . . . . .	58
2.3	The Skeleton of a Polygon . . . . .	58
2.4	Voronoi Diagrams . . . . .	60
2.4.1	Voronoi Diagrams of Families of Closed Sets . . . . .	60
2.4.2	Voronoi Diagrams of Discretized Boundaries . . . . .	61
2.5	Thinning . . . . .	62
2.6	Sensitivity to Noise . . . . .	63
2.7	Recovering the Initial Curve from the MAT . . . . .	64
2.8	Generating Curves from Medial and Skeletal Structures . . . . .	65
2.8.1	Skeletons with Linear Branches . . . . .	66
2.8.2	Skeletal Structures . . . . .	70
<b>3</b>	<b>Local Properties of Surfaces . . . . .</b>	<b>73</b>
3.1	Curves in Three Dimensions . . . . .	73
3.2	Regular Surfaces . . . . .	74
3.2.1	Examples . . . . .	75
3.2.2	Changing Coordinates . . . . .	80
3.2.3	Implicit Surfaces . . . . .	80
3.3	Tangent Planes and Differentials . . . . .	80
3.3.1	Tangent Planes . . . . .	80
3.3.2	Differentials . . . . .	81
3.4	Orientation and Normals . . . . .	83
3.5	Integration on an Orientable Surface . . . . .	84
3.6	Regular Surfaces with Boundary . . . . .	86
3.7	The First Fundamental Form . . . . .	87
3.7.1	Definition and Properties . . . . .	87
3.7.2	Geodesics . . . . .	88
3.7.3	The Divergence Theorem on Surfaces . . . . .	90
3.8	Curvature and the Second Fundamental Form . . . . .	93
3.9	Curvature in Local Coordinates . . . . .	95
3.10	Implicit Surfaces . . . . .	96
3.11	The Gauss–Bonnet Theorem . . . . .	100

- 4 Computations on Triangulated Surfaces** . . . . . 101
  - 4.1 Triangulated Surfaces . . . . . 101
    - 4.1.1 Definition and Notation . . . . . 101
  - 4.2 Estimating the Curvatures . . . . . 102
    - 4.2.1 Taylor Expansions . . . . . 102
    - 4.2.2 Gauss–Bonnet and Area Minimization . . . . . 105
    - 4.2.3 Curvature Measures . . . . . 107
    - 4.2.4 Discrete Gradient and Laplace–Beltrami Operators . . . . . 116
  - 4.3 Consistent Approximation . . . . . 119
  - 4.4 Isocontours and Isosurfaces . . . . . 122
    - 4.4.1 Computing Isocontours . . . . . 123
    - 4.4.2 Computing Isosurfaces . . . . . 125
- 5 Evolving Curves and Surfaces** . . . . . 131
  - 5.1 Curve Evolution . . . . . 131
    - 5.1.1 Grassfires . . . . . 135
    - 5.1.2 Curvature Motion . . . . . 137
    - 5.1.3 Implicit Representation of the Curvature Motion . . . . . 139
    - 5.1.4 More on the Implementation . . . . . 141
  - 5.2 Surface Evolution . . . . . 143
    - 5.2.1 Mean Curvature Surface Flow . . . . . 150
  - 5.3 Gradient Flows . . . . . 150
  - 5.4 Active Contours . . . . . 154
    - 5.4.1 Introduction . . . . . 154
    - 5.4.2 First Variation and Gradient Descent . . . . . 154
    - 5.4.3 Numerical Implementation . . . . . 155
    - 5.4.4 Initialization and Other Driving Techniques . . . . . 157
    - 5.4.5 Evolution of the Parametrization . . . . . 158
    - 5.4.6 Parametrization-Invariant Methods . . . . . 160
    - 5.4.7 Controlled Curve Evolution . . . . . 162
    - 5.4.8 Geometric Active Surfaces . . . . . 162
    - 5.4.9 Designing the  $V$  Function . . . . . 163
    - 5.4.10 Inside/Outside Optimization . . . . . 163
    - 5.4.11 Sobolev Active Contours . . . . . 166
- 6 Deformable Templates** . . . . . 169
  - 6.1 Linear Representations . . . . . 169
    - 6.1.1 Energetic Representation . . . . . 170
  - 6.2 Probabilistic Decompositions . . . . . 173
    - 6.2.1 Deformation Axes . . . . . 173
  - 6.3 Stochastic Deformation Models . . . . . 174
    - 6.3.1 Generalities . . . . . 174

6.3.2	Representation and Deformations of Planar Polygonal Shapes . . . . .	175
6.4	Segmentation with Deformable Templates . . . . .	178
<b>7</b>	<b>Ordinary Differential Equations and Groups of Diffeomorphisms . . . . .</b>	<b>183</b>
7.1	Introduction . . . . .	183
7.2	Flows and Groups of Diffeomorphisms . . . . .	193
7.2.1	Definitions . . . . .	193
7.2.2	Variation in the Initial Condition . . . . .	193
7.2.3	Variations with Respect to the Vector Field . . . . .	195
7.2.4	Weak Continuity . . . . .	196
7.2.5	Admissible Banach Spaces . . . . .	198
7.2.6	A Distance on $G_V$ . . . . .	199
7.2.7	Properties of the Distance . . . . .	201
<b>8</b>	<b>Building Admissible Spaces . . . . .</b>	<b>205</b>
8.1	Reproducing Kernel Hilbert Spaces . . . . .	205
8.1.1	The Scalar Case . . . . .	205
8.1.2	The Vector Case . . . . .	210
8.1.3	Derivatives of the Kernel . . . . .	213
8.2	Building $V$ from Operators . . . . .	214
8.3	Invariance of the Inner Product . . . . .	220
8.3.1	Invariance: The Operator Side . . . . .	220
8.3.2	Invariance: The Kernel Side . . . . .	230
8.4	Mercer's Theorem . . . . .	235
8.5	Thin-Plate Interpolation . . . . .	237
8.6	Asymptotically Affine Kernels . . . . .	239
<b>9</b>	<b>Deformable Objects and Matching Functionals . . . . .</b>	<b>243</b>
9.1	General Principles . . . . .	243
9.2	Differentiation with Respect to Diffeomorphisms . . . . .	244
9.3	Relation with Matching Functionals . . . . .	246
9.4	Labeled Point Matching . . . . .	247
9.5	Image Matching . . . . .	249
9.6	Measure Matching . . . . .	255
9.6.1	Matching Densities . . . . .	258
9.6.2	Dual RKHS Norms on Measures . . . . .	259
9.7	Matching Curves and Surfaces . . . . .	262
9.7.1	Curve Matching with Measures . . . . .	263
9.7.2	Curve Matching with Vector Measures . . . . .	267
9.7.3	Surface Matching . . . . .	268
9.7.4	Induced Actions and Currents . . . . .	272
9.7.5	Varifolds . . . . .	276

- 9.8 Matching Vector Fields . . . . . 277
- 9.9 Matching Fields of Frames . . . . . 282
- 9.10 Matching Tensors . . . . . 285
- 9.11 Pros and Cons of Greedy Algorithms . . . . . 289
- 10 Diffeomorphic Matching . . . . . 291**
  - 10.1 Linearized Deformations . . . . . 291
  - 10.2 The Monge–Kantorovitch Problem . . . . . 293
  - 10.3 Optimizing Over Flows . . . . . 296
  - 10.4 Euler–Lagrange Equations and Gradient . . . . . 298
    - 10.4.1 Gradient: Direct Computation . . . . . 298
    - 10.4.2 Derivative Using Optimal Control . . . . . 300
    - 10.4.3 An Alternative Form Using the RKHS Structure . . . . . 301
  - 10.5 Conservation of Momentum . . . . . 302
    - 10.5.1 Interpretation . . . . . 302
    - 10.5.2 Properties of the Momentum Conservation Equation . . . . . 304
    - 10.5.3 Time Variation of the Eulerian Momentum . . . . . 309
    - 10.5.4 Explicit Expression . . . . . 311
    - 10.5.5 The Hamiltonian Form of EPDiff . . . . . 311
    - 10.5.6 The Case of Measure Momenta . . . . . 312
  - 10.6 Optimization Strategies for Flow-Based Matching . . . . . 314
    - 10.6.1 Gradient Descent in  $\mathcal{X}_V^2$  . . . . . 314
    - 10.6.2 Gradient in the Hamiltonian Form . . . . . 318
    - 10.6.3 Gradient in the Initial Momentum . . . . . 322
    - 10.6.4 Shooting . . . . . 328
    - 10.6.5 Gradient in the Deformable Object . . . . . 330
    - 10.6.6 Image Matching . . . . . 333
    - 10.6.7 Pros and Cons of the Optimization Strategies . . . . . 337
  - 10.7 Numerical Aspects . . . . . 338
    - 10.7.1 Discretization . . . . . 338
    - 10.7.2 Kernel-Related Numerics . . . . . 344
- 11 Distances and Group Actions . . . . . 347**
  - 11.1 General Principles . . . . . 347
    - 11.1.1 Distance Induced by a Group Action . . . . . 347
    - 11.1.2 Distance Altered by a Group Action . . . . . 349
    - 11.1.3 Transitive Action . . . . . 349
    - 11.1.4 The Riemannian Viewpoint . . . . . 352
  - 11.2 Invariant Distances Between Point Sets . . . . . 353
    - 11.2.1 Introduction . . . . . 353
    - 11.2.2 The Space of Planar  $N$ -Shapes . . . . . 353
  - 11.3 Parametrization-Invariant Distances Between Plane Curves . . . . . 357

11.4	Invariant Metrics on Diffeomorphisms . . . . .	360
11.4.1	The Geodesic Equation . . . . .	361
11.4.2	A Simple Example . . . . .	362
11.4.3	Gradient Descent . . . . .	363
11.4.4	Diffeomorphic Active Contours . . . . .	364
11.5	Group Actions and Riemannian Submersion . . . . .	365
11.5.1	Riemannian Submersion . . . . .	365
11.5.2	The Momentum Representation . . . . .	368
<b>12</b>	<b>Metamorphosis . . . . .</b>	<b>373</b>
12.1	Definitions . . . . .	373
12.2	A New Metric on $M$ . . . . .	374
12.3	Euler–Lagrange Equations . . . . .	377
12.4	Application to Labeled Point Sets . . . . .	378
12.5	Application to Images . . . . .	380
12.5.1	Formal Analysis . . . . .	380
12.5.2	Some Rigorous Results . . . . .	382
12.5.3	Remarks on the Optimality Conditions . . . . .	384
12.5.4	Discretization of Image Metamorphosis in the $L^2$ Case . . . . .	388
12.6	Applications to Densities . . . . .	388
12.7	Application to Curves . . . . .	389
12.7.1	Metamorphosis on Unit Tangents . . . . .	389
12.7.2	One-Dimensional Case . . . . .	396
12.7.3	The Smooth Case . . . . .	397
12.7.4	Existence of Optimal Metamorphoses . . . . .	398
12.7.5	The Case of Closed Curves . . . . .	400
12.7.6	Alternative Interpretation ( $\sigma = 1$ ) . . . . .	403
<b>13</b>	<b>Analyzing Shape Datasets . . . . .</b>	<b>405</b>
13.1	Reference-Based Representation . . . . .	405
13.2	Principal Component Analysis . . . . .	409
13.3	Time Series . . . . .	411
13.3.1	Single Trajectory . . . . .	411
13.3.2	Multiple Trajectories . . . . .	416
13.3.3	Reference-Centered Representations of Time Series . . . . .	419
<b>Appendix A: Elements from Functional Analysis . . . . .</b>		<b>423</b>
<b>Appendix B: Elements from Differential Geometry . . . . .</b>		<b>439</b>
<b>Appendix C: Ordinary Differential Equations . . . . .</b>		<b>481</b>
<b>Appendix D: Introduction to Optimization and Optimal Control Theory . . . . .</b>		<b>505</b>

<b>Appendix E: Principal Component Analysis</b> . . . . .	527
<b>Appendix F: Dynamic Programming</b> . . . . .	533
<b>References</b> . . . . .	541
<b>Index</b> . . . . .	555



# List of Figures

Fig. 1.1 Noise and curvature. The first curve on the left is an ellipse discretized over 125 points. The second on the right is the same ellipse, with coordinates rounded to two decimal points. The difference is almost imperceptible. However, the second row shows the result of estimating the curvature without smoothing, on the first and the second ellipse, with a very strong noise effect. The third (resp. fourth) row shows the result of the second-order approximation with  $\Delta = 5$  (resp.  $\Delta = 10$ ). The computed curvature for the truncated curve progressively improves while that of the original curve is minimally affected. . . . . 22

Fig. 1.2 Implicit representation: The first two panels provide two views of the graph of a function  $f : \mathbb{R}^2 \mapsto \mathbb{R}$  intersecting the plane  $z = 0$ . The third panel is the corresponding level set . . . . . 29

Fig. 1.3 Shapes generated by implicit polynomials of degree 4. The first curve is the level set of the polynomial  $f(x, y) = x^4 + y^4 - xy - 0.1$ . The other three are generated by adding a small noise to the coefficients (including zeros) of  $f$ . . . . . 31

Fig. 1.4 Conformal disparity between the interior and exterior of four planar curves. *First column*: original curves; *second and third columns*: two representations of the curve signature rescaled over the unit interval, related by a Möbius transformation, illustrating the fact that these signatures are equivalent classes of diffeomorphisms of the unit disc . . . . . 53

Fig. 1.5 Reconstruction of the curves in Fig. 1.4 from their signatures . . . . . 54

Fig. 2.1 Comparison of medial axes computed using Voronoï diagrams with different degrees of discretization . . . . . 62

Fig. 2.2 Effect of a small shape change in the boundary on the skeleton of a rectangular shape. . . . . 63

Fig. 2.3 Shapes with horizontal medial axes. The shapes (right column) are obtained with  $\partial_{uu}^2(r^2) = -2f$ ;  $f$  is shown in the left column . . . . . 68

Fig. 2.4 Shapes generated from a medial axis with three linear branches . . . . . 70

Fig. 3.1 Helicoid (left) and cylinder (right) . . . . . 76

Fig. 3.2 Sphere (left) and torus (right) . . . . . 77

Fig. 3.3 Examples of decompositions in local charts for the sphere and the torus. Parametrizations are represented by grids over the surface, black inside local patches and gray outside . . . . . 78

Fig. 3.4 A non-regular surface . . . . . 79

Fig. 3.5 Two examples of non-orientable surfaces. On the left is the Möbius band; the surface on the right is similar, with an odd number of twists. . . . . 84

Fig. 3.6 The Gauss–Bonnet theorem in  $\mathbb{R}^2$  reduces to the well-known property that the sum of consecutive angles in a polygon is  $2\pi$ . . . . . 100

Fig. 4.1 Decomposition of triangles into Voronoï cells when the circumcenter is interior to the triangle (left) and when it is exterior (right) . . . . . 103

Fig. 4.2 Isocontouring a checkerboard strip using exact bilinear rule (d) (first row) and sign-based rule (d)' (second row). Note that the solutions are different, although both are plausible isocontours for the image. Gray levels are switched in the last two rows, without changing the solution for rule (d) (third row) and significantly altering it for rule (d)' (fourth row), yielding a third plausible solution. . . . . 126

Fig. 4.3 Two-component (non-ambiguous) cases for the marching cubes algorithm . . . . . 127

Fig. 4.4 Labels for the vertices (left) and edges (right) of the cube . . . . . 128

Fig. 4.5 Two triangulations associated to the  $\{3, 8\}$  sign configuration  $[(2, 3, 11), (1, 4, 9)]$  and  $[(4, 9, 3), (3, 9, 11), (9, 11, 2), (1, 2, 9)]$ . . . . . 128

Fig. 4.6 Two triangulations associated to the  $\{1, 7\}$  sign configuration  $[(1, 4, 9), (7, 6, 11)]$  and  $[(1, 4, 11), (1, 11, 6), (1, 9, 6), (9, 6, 7), (9, 4, 7), (4, 11, 7)]$  . . . . . 129

Fig. 4.7 Twenty-three configurations for consistent within-cube triangulation based on vertex signs. Dotted vertices correspond to positive values of the function . . . . . 129

Fig. 5.1 Curve evolution with the grassfire equation. Left: evolution according to (5.6). Right: same evolution after removing the burned-out parts . . . . . 137

Fig. 5.2 Curves evolving according to the curve-shortening flow. *First row*: three examples of superimposed evolving shapes. *Second row*: details of the evolution of the third shape (spiral) in the first row . . . . . 140

Fig. 5.3 Signed distance map. *Left*: original curve. *Right*: associated signed distance map . . . . . 141

Fig. 5.4 Comparison between geometric and Sobolev active contours. In each row, the left image is the initial contour for both algorithms, the center one is the result of geometric active contours once the evolution stabilizes, and the right one is for Sobolev active contours. The first row presents a clean image and the second a noisy one. In both cases, geometric active contours get stuck at an unsatisfactory local minimum (especially in the noisy case), while Sobolev active contours reach a much more accurate solution . . . . . 167

Fig. 6.1 Random deformations of a circle (with different values for  $\alpha$  and  $\beta$ ) . . . . . 178

Fig. 9.1 Greedy landmark matching. Implementation of the gradient descent algorithm in (9.14), starting with  $\varphi = \text{id}$ , for the correspondences depicted in the upper-left image (diamonds moving to circles). The following three images provide the result after numerical convergence for Gaussian kernels  $K(x, y) = \exp(-|x - y|^2/2\sigma^2)\text{Id}$  with  $\sigma = 1, 2, 4$  in grid units. Larger  $\sigma$  induce increasing smoothness in the final solution, and deformations affecting a larger part of the space. As seen in the figure for  $\sigma = 4$ , the evolution can result in huge deformations . . . . . 250

Fig. 9.2 Greedy image matching. Output of Algorithm 9.5 when estimating a deformation of the first image to match the second one. The third image is the obtained deformation of the first one and the last provides the deformation applied to a grid . . . . . 254

Fig. 10.1 Metric point matching. The first two rows provide results obtained with gradient descent in the initial momentum for point matching, with the same input as in Fig. 9.1, using Gaussian kernels  $K(x, y) = \exp(-|x - y|^2/2\sigma^2)$  with  $\sigma = 1, 2, 4$  in grid units. The impact of the diffeomorphic regularization on the quality of the result is particularly obvious in the last experiment. The last row provides the output of Gaussian spline registration with the same kernels, exhibiting singularities and ambiguities in the registration . . . . . 329

Fig. 10.2 Metric image matching. Output of Algorithm 4 when estimating a deformation of the first image to match the second one (compare to Fig. 9.2). The third image is the obtained deformation of the first one and the last provides the deformation applied to a grid . . . . . 336

Fig. 11.1 Diffeomorphic active contours (compare with Fig. 5.4). On each row, the left image is the initial contour, and the right one is the solution obtained with diffeomorphic active contours. The first row presents a clean image and the second a noisy one . . . . . 365

Fig. 12.1 Point-set matching using metamorphosis, with the same input as in Figs. 9.1 and 10.1, using Gaussian kernels  $K(x, y) = \exp(-|x - y|^2/2\sigma^2)$  with  $\sigma = 1, 2, 4$  in grid units . . . . . 381

Fig. 12.2 Image metamorphosis. Estimation of a geodesic between a disc and a square. First row: image evolution in time. Second row: evolution of the diffeomorphism . . . . . 382

Fig. 12.3 Creation of new patterns with image metamorphosis. Geodesic between an empty image and a disc. First row: image evolution in time. Second row: evolution of the diffeomorphism . . . . . 382

Fig. 12.4 Gray-level image metamorphosis. Geodesic between two gray-level face images. First row: image evolution in time. Second row: evolution of the diffeomorphism . . . . . 383

Fig. 12.5 Two sets of examples of optimal metamorphoses between curves with  $\sigma = 0.5$  (left), 1 (center) and 2 (right). Note that, in the first row, the geodesics with  $\sigma = 0.5$  and  $\sigma = 1$  grow the missing finger out of the second finger, while that with  $\sigma = 2$  grows a new thumb. In each image, the geodesic is computed between the outer curve and the inner curve. Scaling is for visualization only (the computation is scale-invariant) . . . . . 399

Fig. 12.6 Optimal correspondences estimated with the three metamorphoses in Fig. 12.5. Background images represent the function  $\max(f_\sigma, 0)$  in (12.45), the zeros being indicated in black. The solution attempts to move across higher values of the function, cutting through zeros along vertical or horizontal lines if needed. Each panel corresponds to the same panel in Fig. 12.5 . . . . . 400

Fig. 13.1 Example of template estimation. Rows 2 and 3: ten randomly generated shapes. Row 1: hypertemplate, chosen as the first shape in the sequence. Row 4: estimated template . . . . . 407

Fig. 13.2 Time series with four surfaces: left to right:  $t = 1, 2, 3, 4$ . . . . . 415

Fig. 13.3 Piecewise geodesic interpolation of sequence in Fig. 13.2 with seven time points (left to right and top to bottom):  $t = 1, 1.5, 2, 2.5, 3, 3.5, 4$ . . . . . 415

Fig. 13.4 Spline interpolation of sequence in Fig. 13.2 with seven time points (left to right and top to bottom):  
 $t = 1, 1.5, 2, 2.5, 3, 3.5, 4$  . . . . . 415

Fig. 13.5 Geodesic interpolation of sequence in Fig. 13.2 with seven time points (left to right and top to bottom):  
 $t = 1, 1.5, 2, 2.5, 3, 3.5, 4$  . . . . . 416

Fig. 13.6 Evolution of the volumes of the interpolated surfaces in Fig. 13.2. The first leg of the trajectory (from time 0 to 1) is the adjustment of the baseline starting with the volume of the “hypertemplate”. The dots that follow are the volumes of the target surfaces. The time interpolation step is  $\delta t = 0.1$ . From left to right: piecewise geodesic, spline and geodesic . . . . . 416

Fig. F.1 A typical distance map in 1D, followed by a discretization over an odd and even number of points . . . . . 537

Fig. F.2 A curve and its distance map . . . . . 540