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Florian Scheck

Classical Field Theory

On Electrodynamics, Non-Abelian Gauge Theories and Gravitation

Second Edition

Springer

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To Dörte

Preface

Traditionally one begins a course or a textbook on electrodynamics with an extensive discussion of electrostatics, of magnetostatics, and of stationary currents, before turning to the full time-dependent Maxwell theory in local form. In this book, I choose a somewhat different approach: Starting from Maxwell's equations in integral form, that is to say, from the phenomenological and experimentally verified basis of electrodynamics, the local equations are formulated and discussed with their general time and space dependence right from the start. Static or stationary situations appear as special cases for which Maxwell's equations split into two more or less independent groups and thus are decoupled to a certain extent.

Great importance is attached to the symmetries of the Maxwell equations and, in particular, their covariance with respect to Lorentz transformations. Another central issue is the treatment of electrodynamics in the framework of classical field theory by means of a Lagrange density and Hamilton's principle. General principles that were developed for mechanics, appear in a deeper and more general application that can serve as a model and prototype for any classical field theory. The fact that the fields of Maxwell theory, in general, depend on space *and* time makes it necessary to enlarge the framework of traditional tensor analysis in $\mathbb{R}^3$ to exterior calculus on $\mathbb{R}^4$. The venerable vector and tensor analysis that was designed for three-dimensional Euclidean spaces, does not suffice and must be generalized to higher dimensions and to Minkowski signature. While the exterior product is the generalization of the vector product in $\mathbb{R}^3$, Cartan's exterior derivative is the natural generalization of the curl in $\mathbb{R}^3$ and, by the same token, encompasses the familiar operations of gradient and divergence.

Among the many applications of Maxwell theory I chose some characteristic and, I felt, nowadays particularly relevant examples such as an extensive discussion of polarization of electromagnetic waves, the description of Gaussian beams (these are analytic solutions of the Helmholtz equation in paraxial approximation), and optics of meta-materials with negative index of refraction. Regarding other, more traditional applications I refer to the well-known, excellent textbooks by J.D. Jackson, by L.D. Landau and E.M. Lifshits, and others.

As a novel feature I take up in the fifth chapter, a further direction of great importance for present-day physics: The construction of non-Abelian gauge theories. These Yang–Mills theories as they are called[1], are essential and indispensable for our present understanding of the fundamental interactions of nature. Although these theories which are at the basis of the so-called standard model of elementary particle physics, lead us far into *quantized* field theory, their construction and their essential features are of a *classical* nature, at least as long as one considers only the radiation fields, the analogues of the Maxwell fields, and classical scalar fields, but leaves out fermionic matter particles. Non-Abelian gauge theories are constructed following the example of Maxwell theory. They bear some similarities to the latter, but exhibit also significant differences from it. Even the phenomenon of spontaneous symmetry breaking that preserves us from the appearance of too many massless fields, in essence, is a classical mechanism. In view of the great impact of gauge theories on our understanding of the fundamental interactions, it would be a loss not to do this step which builds on Maxwell theory in a most natural manner.

Chapter 6 gives an extensive phenomenological and geometric introduction to general relativity and, hence, rounds off the description of all fundamental interactions in the framework of classical field theory. Here too, I use consistently a modern geometric language which—after some investment in differential geometry—allows for a transparent formulation of Einstein's equations which is better focused to its essentials than the older tensor analysis formulated in components only. The present second edition was enlarged and revised in several respects. The most important additions were made in Chap. 6 to which I added the derivation of the Schwarzschild solution in the framework of Cartan's structure equations, the Kruskal continuation of this solution to the interior of the Schwarzschild radius, a discussion of black holes and of rotating solutions and, finally, a description of gravitational waves.

Much of the material included in this book was tried out in numerous lectures that I gave at Johannes Gutenberg University over the years. I am grateful to the students who have followed these courses, for their questions and comments, and to the teaching assistants who took good care of exercise classes, for their stimulating questions and critical comments.

I owe special thanks to Immanuel Bloch for the discussions we had on Gaussian beams and the fascinating topic of meta-materials with negative index of refraction, and for his encouragement to include these modern applications. Special thanks also Mario Paschke for stimulating discussions on original ideas and for hints at some older but relevant references, as well as to Andrès Reyes-Lega for numerous discussions over the years and for careful reading of major parts of the book. I am also very grateful to Maximilian Becker who read very carefully several parts of the book, notably the new sections in Chap. 6, and made numerous suggestions for their presentation.

[1]First ideas were published by Oskar Klein, Z. Physik 37 (1926) 895. It is reported that Wolfgang Pauli developed them independently but did not publish them.

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Mainz, Germany Florian Scheck
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