

Mathematical Engineering

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Günter Brenn

Analytical Solutions for Transport Processes

Fluid Mechanics, Heat and Mass Transfer

 Springer

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ISSN 2192-4732

Mathematical Engineering

ISBN 978-3-662-51421-4

DOI 10.1007/978-3-662-51423-8

ISSN 2192-4740 (electronic)

ISBN 978-3-662-51423-8 (eBook)

Library of Congress Control Number: 2016940362

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*This book is dedicated to Micha
and Daniel for their patience and support*

Preface

The availability of increasing computer power and efficient numerical methods for the simulation of transport processes have led to an increased use of—in part commercial—computational tools in science and engineering. The applications range from electrodynamics to fluid mechanics, from production technology to the life sciences. In many fields and cases, however, the interpretation of computational results remains difficult, since relevant details of the (commercial) computer codes are often unknown. One possible way to test the ability of a code to solve a given problem or to reproduce a physical effect is to run it for a related classical problem with a known analytical solution. A correct reproduction of the analytical solution enhances the confidence in the numerical code. For the purpose of benchmarking computer codes, the analytical solutions of classical problems are therefore of high value.

Another value of analytical solutions is that they provide an understanding of the physical phenomena underlying a given process, including the trends of the behaviour of a function for very low and/or very high values of its argument(s). An analytical equation representing the behaviour of a solution by elementary functions is of higher value than columns of numbers that do not immediately show the underlying functional dependencies.

The restrictions underlying analytical solutions, on the other hand, are manifold. Typically, analytical solutions of transport processes are possible for two-dimensional laminar flow in simple geometries only, where the material properties are treated as constants. This aspect, however, does not put the value of analytical solutions in question altogether.

The present book presents and discusses analytical solutions of problems in transport processes, i.e. in fluid mechanics and in heat and mass transfer, with the intention to give an overview of the field, with some selected in-depth analyses. The problems and their solutions are in a bigger part classical, but complemented by results from recent research. Starting from the underlying equations of change of continuum mechanics and thermodynamics, the derivations of the equations and their solutions address all the simplifications and assumptions involved and

quantify the functional dependencies of the resulting quantities on the influencing variables and parameters. The book includes appendices listing the equations of change of continuum mechanics and thermodynamics, important vector analytical operations with scalar and vectorial functions encountered in transport processes, and selected special functions of mathematical physics used in the analyses presented in the book.

The book does not claim completeness in any respect, but rather puts together classical methods and solutions for transport processes in a widest possible field, with an emphasis on multiphase flows under the influence of capillary forces. The selection results from the research interests of the author.

The book addresses readers in research and development working in the field of the physics of transport processes, early-stage researchers entering the field and users of computational tools to be applied to transport problems in science and technology.

Graz
March 2016

Günter Brenn

Acknowledgements

In writing this book, the author was supported in various respects by a number of persons. The author is indebted to Prof. Dr. A.L. Yarin at the University of Illinois in Chicago for innumerable valuable pieces of advice in writing this book and for inspections of the manuscript in different stages of its development. Valuable discussions with my colleague ao. Univ.-Prof. Dipl.-Ing. Dr. techn. H. Steiner on problems discussed in the book helped significantly to improve their descriptions. A number of data sets displayed in diagrams were produced by him. The support from univ. dipl. fiz. Gregor Plohl, assistant of our Institute of Fluid Mechanics and Heat Transfer, by producing data displayed in diagrams is gratefully acknowledged. The author gratefully acknowledges the assistance in preparing the figures from Andreas Egger, B.Sc., master student of Mechanical Engineering of our department at Graz University of Technology. Also, the author is grateful to Mrs. Sabine Gruber, secretary of our institute, for typesetting the equations of change in the Appendix. Without these persons, this book would not have been finished.

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Acronyms

The present list of abbreviations puts together acronyms used in the book to enhance the readability of the text.

LDA	Laser-Doppler Anemometry
ODE	Ordinary differential equation
PDA	Phase-Doppler Anemometry
PDE	Partial differential equation