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(continued after index)

Tom M. Apostol

Introduction to Analytic Number Theory

With 24 Illustrations



Springer

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Preface

This is the first volume of a two-volume textbook¹ which evolved from a course (Mathematics 160) offered at the California Institute of Technology during the last 25 years. It provides an introduction to analytic number theory suitable for undergraduates with some background in advanced calculus, but with no previous knowledge of number theory. Actually, a great deal of the book requires no calculus at all and could profitably be studied by sophisticated high school students.

Number theory is such a vast and rich field that a one-year course cannot do justice to all its parts. The choice of topics included here is intended to provide some variety and some depth. Problems which have fascinated generations of professional and amateur mathematicians are discussed together with some of the techniques for solving them.

One of the goals of this course has been to nurture the intrinsic interest that many young mathematics students seem to have in number theory and to open some doors for them to the current periodical literature. It has been gratifying to note that many of the students who have taken this course during the past 25 years have become professional mathematicians, and some have made notable contributions of their own to number theory. To all of them this book is dedicated.

¹ The second volume is scheduled to appear in the Springer-Verlag Series Graduate Texts in Mathematics under the title **Modular Functions and Dirichlet Series in Number Theory**.

Contents

Historical Introduction

Chapter 1

The Fundamental Theorem of Arithmetic

- 1.1 Introduction 13
- 1.2 Divisibility 14
- 1.3 Greatest common divisor 14
- 1.4 Prime numbers 16
- 1.5 The fundamental theorem of arithmetic 17
- 1.6 The series of reciprocals of the primes 18
- 1.7 The Euclidean algorithm 19
- 1.8 The greatest common divisor of more than two numbers 20
- Exercises for Chapter 1 21*

Chapter 2

Arithmetical Functions and Dirichlet Multiplication

- 2.1 Introduction 24
- 2.2 The Möbius function $\mu(n)$ 24
- 2.3 The Euler totient function $\varphi(n)$ 25
- 2.4 A relation connecting φ and μ 26
- 2.5 A product formula for $\varphi(n)$ 27
- 2.6 The Dirichlet product of arithmetical functions 29
- 2.7 Dirichlet inverses and the Möbius inversion formula 30
- 2.8 The Mangoldt function $\Lambda(n)$ 32
- 2.9 Multiplicative functions 33
- 2.10 Multiplicative functions and Dirichlet multiplication 35
- 2.11 The inverse of a completely multiplicative function 36

- 2.12 Liouville's function $\lambda(n)$ 37
- 2.13 The divisor functions $\sigma_x(n)$ 38
- 2.14 Generalized convolutions 39
- 2.15 Formal power series 41
- 2.16 The Bell series of an arithmetical function 42
- 2.17 Bell series and Dirichlet multiplication 44
- 2.18 Derivatives of arithmetical functions 45
- 2.19 The Selberg identity 46
- Exercises for Chapter 2* 46

Chapter 3

Averages of Arithmetical Functions

- 3.1 Introduction 52
- 3.2 The big oh notation. Asymptotic equality of functions 53
- 3.3 Euler's summation formula 54
- 3.4 Some elementary asymptotic formulas 55
- 3.5 The average order of $d(n)$ 57
- 3.6 The average order of the divisor functions $\sigma_x(n)$ 60
- 3.7 The average order of $\varphi(n)$ 61
- 3.8 An application to the distribution of lattice points visible from the origin 62
- 3.9 The average order of $\mu(n)$ and of $\Lambda(n)$ 64
- 3.10 The partial sums of a Dirichlet product 65
- 3.11 Applications to $\mu(n)$ and $\Lambda(n)$ 66
- 3.12 Another identity for the partial sums of a Dirichlet product 69
- Exercises for Chapter 3* 70

Chapter 4

Some Elementary Theorems on the Distribution of Prime Numbers

- 4.1 Introduction 74
- 4.2 Chebyshev's functions $\psi(x)$ and $\vartheta(x)$ 75
- 4.3 Relations connecting $\vartheta(x)$ and $\pi(x)$ 76
- 4.4 Some equivalent forms of the prime number theorem 79
- 4.5 Inequalities for $\pi(n)$ and p_n 82
- 4.6 Shapiro's Tauberian theorem 85
- 4.7 Applications of Shapiro's theorem 88
- 4.8 An asymptotic formula for the partial sums $\sum_{p \leq x} (1/p)$ 89
- 4.9 The partial sums of the Möbius function 91
- 4.10 Brief sketch of an elementary proof of the prime number theorem 98
- 4.11 Selberg's asymptotic formula 99
- Exercises for Chapter 4* 101

Chapter 5

Congruences

- 5.1 Definition and basic properties of congruences 106
- 5.2 Residue classes and complete residue systems 109
- 5.3 Linear congruences 110

- 5.4 Reduced residue systems and the Euler–Fermat theorem 113
- 5.5 Polynomial congruences modulo p . Lagrange’s theorem 114
- 5.6 Applications of Lagrange’s theorem 115
- 5.7 Simultaneous linear congruences. The Chinese remainder theorem 117
- 5.8 Applications of the Chinese remainder theorem 118
- 5.9 Polynomial congruences with prime power moduli 120
- 5.10 The principle of cross-classification 123
- 5.11 A decomposition property of reduced residue systems 125
- Exercises for Chapter 5* 126

Chapter 6

Finite Abelian Groups and Their Characters

- 6.1 Definitions 129
- 6.2 Examples of groups and subgroups 130
- 6.3 Elementary properties of groups 130
- 6.4 Construction of subgroups 131
- 6.5 Characters of finite abelian groups 133
- 6.6 The character group 135
- 6.7 The orthogonality relations for characters 136
- 6.8 Dirichlet characters 137
- 6.9 Sums involving Dirichlet characters 140
- 6.10 The nonvanishing of $L(1, \chi)$ for real nonprincipal χ 141
- Exercises for Chapter 6* 143

Chapter 7

Dirichlet’s Theorem on Primes in Arithmetic Progressions

- 7.1 Introduction 146
- 7.2 Dirichlet’s theorem for primes of the form $4n - 1$ and $4n + 1$ 147
- 7.3 The plan of the proof of Dirichlet’s theorem 148
- 7.4 Proof of Lemma 7.4 150
- 7.5 Proof of Lemma 7.5 151
- 7.6 Proof of Lemma 7.6 152
- 7.7 Proof of Lemma 7.8 153
- 7.8 Proof of Lemma 7.7 153
- 7.9 Distribution of primes in arithmetic progressions 154
- Exercises for Chapter 7* 155

Chapter 8

Periodic Arithmetical Functions and Gauss Sums

- 8.1 Functions periodic modulo k 157
- 8.2 Existence of finite Fourier series for periodic arithmetical functions 158
- 8.3 Ramanujan’s sum and generalizations 160
- 8.4 Multiplicative properties of the sums $s_k(n)$ 162
- 8.5 Gauss sums associated with Dirichlet characters 165
- 8.6 Dirichlet characters with nonvanishing Gauss sums 166
- 8.7 Induced moduli and primitive characters 167

- 8.8 Further properties of induced moduli 168
- 8.9 The conductor of a character 171
- 8.10 Primitive characters and separable Gauss sums 171
- 8.11 The finite Fourier series of the Dirichlet characters 172
- 8.12 Pólya's inequality for the partial sums of primitive characters 173
- Exercises for Chapter 8* 175

Chapter 9

Quadratic Residues and the Quadratic Reciprocity Law

- 9.1 Quadratic residues 178
- 9.2 Legendre's symbol and its properties 179
- 9.3 Evaluation of $(-1|p)$ and $(2|p)$ 181
- 9.4 Gauss' lemma 182
- 9.5 The quadratic reciprocity law 185
- 9.6 Applications of the reciprocity law 186
- 9.7 The Jacobi symbol 187
- 9.8 Applications to Diophantine equations 190
- 9.9 Gauss sums and the quadratic reciprocity law 192
- 9.10 The reciprocity law for quadratic Gauss sums 195
- 9.11 Another proof of the quadratic reciprocity law 200
- Exercises for Chapter 9* 201

Chapter 10

Primitive Roots

- 10.1 The exponent of a number mod m . Primitive roots 204
- 10.2 Primitive roots and reduced residue systems 205
- 10.3 The nonexistence of primitive roots mod 2^α for $\alpha \geq 3$ 206
- 10.4 The existence of primitive roots mod p for odd primes p 206
- 10.5 Primitive roots and quadratic residues 208
- 10.6 The existence of primitive roots mod p^α 208
- 10.7 The existence of primitive roots mod $2p^\alpha$ 210
- 10.8 The nonexistence of primitive roots in the remaining cases 211
- 10.9 The number of primitive roots mod m 212
- 10.10 The index calculus 213
- 10.11 Primitive roots and Dirichlet characters 218
- 10.12 Real-valued Dirichlet characters mod p^α 220
- 10.13 Primitive Dirichlet characters mod p^α 221
- Exercises for Chapter 10* 222

Chapter 11

Dirichlet Series and Euler Products

- 11.1 Introduction 224
- 11.2 The half-plane of absolute convergence of a Dirichlet series 225
- 11.3 The function defined by a Dirichlet series 226

11.4	Multiplication of Dirichlet series	228
11.5	Euler products	230
11.6	The half-plane of convergence of a Dirichlet series	232
11.7	Analytic properties of Dirichlet series	234
11.8	Dirichlet series with nonnegative coefficients	236
11.9	Dirichlet series expressed as exponentials of Dirichlet series	238
11.10	Mean value formulas for Dirichlet series	240
11.11	An integral formula for the coefficients of a Dirichlet series	242
11.12	An integral formula for the partial sums of a Dirichlet series	243
	<i>Exercises for Chapter 11</i>	246

Chapter 12

The Functions $\zeta(s)$ and $L(s, \chi)$

12.1	Introduction	249
12.2	Properties of the gamma function	250
12.3	Integral representation for the Hurwitz zeta function	251
12.4	A contour integral representation for the Hurwitz zeta function	253
12.5	The analytic continuation of the Hurwitz zeta function	254
12.6	Analytic continuation of $\zeta(s)$ and $L(s, \chi)$	255
12.7	Hurwitz's formula for $\zeta(s, a)$	256
12.8	The functional equation for the Riemann zeta function	259
12.9	A functional equation for the Hurwitz zeta function	261
12.10	The functional equation for L -functions	261
12.11	Evaluation of $\zeta(-n, a)$	264
12.12	Properties of Bernoulli numbers and Bernoulli polynomials	265
12.13	Formulas for $L(0, \chi)$	268
12.14	Approximation of $\zeta(s, a)$ by finite sums	268
12.15	Inequalities for $ \zeta(s, a) $	270
12.16	Inequalities for $ \zeta(s) $ and $ L(s, \chi) $	272
	<i>Exercises for Chapter 12</i>	273

Chapter 13

Analytic Proof of the Prime Number Theorem

13.1	The plan of the proof	278
13.2	Lemmas	279
13.3	A contour integral representation for $\psi_1(x)/x^2$	283
13.4	Upper bounds for $ \zeta(s) $ and $ \zeta'(s) $ near the line $\sigma = 1$	284
13.5	The nonvanishing of $\zeta(s)$ on the line $\sigma = 1$	286
13.6	Inequalities for $ 1/\zeta(s) $ and $ \zeta'(s)/\zeta(s) $	287
13.7	Completion of the proof of the prime number theorem	289
13.8	Zero-free regions for $\zeta(s)$	291
13.9	The Riemann hypothesis	293
13.10	Application to the divisor function	294
13.11	Application to Euler's totient	297
13.12	Extension of Pólya's inequality for character sums	299
	<i>Exercises for Chapter 13</i>	300

Chapter 14

Partitions

- 14.1 Introduction 304
- 14.2 Geometric representation of partitions 307
- 14.3 Generating functions for partitions 308
- 14.4 Euler's pentagonal-number theorem 311
- 14.5 Combinatorial proof of Euler's pentagonal-number theorem 313
- 14.6 Euler's recursion formula for $p(n)$ 315
- 14.7 An upper bound for $p(n)$ 316
- 14.8 Jacobi's triple product identity 318
- 14.9 Consequences of Jacobi's identity 321
- 14.10 Logarithmic differentiation of generating functions 322
- 14.11 The partition identities of Ramanujan 324
- Exercises for Chapter 14* 325

Bibliography 329

Index of Special Symbols 333

Index 335