

# Theory and Applications of Viscous Fluid Flows

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# Theory and Applications of Viscous Fluid Flows

With 80 Figures



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# Preface

This book is the natural sequel to the study of nonviscous fluid flows presented in our recent book entitled “Theory and Applications of Nonviscous Fluid Flows” and published in 2002 by the Physics Editorial Department of Springer-Verlag (ISBN 3-540-41412-6 Springer-Verlag, Berlin, Heidelberg, New York).

The physical concept of viscosity (for so-called “real fluids”) is associated with both incompressible and compressible fluids. Consequently, we have a vast field of theoretical study and applications from which any subsection could have itself provided an area for a single book. It was, however, decided to attempt a global study so that each chapter serves as an introduction to more specialized study, and the book as a whole presents a necessary broad foundation for further study in depth. Consequently, this volume contains many more pages than my preceding book devoted to nonviscous fluid flows and a large number (80) of figures.

There are *three main models* for the study of viscous fluid flows: First, the model linked with *viscous incompressible fluid flows*, the so-called (dynamic) *Navier model*, governing linearly viscous divergenceless and homogeneous fluid flows.

The second is the so-called *Navier–Stokes model* (NS) which is linked to *compressible*, linearly *viscous and isentropic* equations for a polytropic viscous gas.

The third is the so-called *Navier–Stokes–Fourier* model (NSF) that governs the motion of a *compressible*, linearly *viscous, heat-conducting* gas.

The book has been written for final year undergraduates, graduates, postgraduate research workers and for young researchers in fluid mechanics, applied mathematics, and theoretical/mathematical physics, who are interested in a rational and systematic account of theoretical aspects of viscous fluid flows phenomena and modeling them in relation to practical viscous and heat-conducting problems.

The book is divided into an Introduction and ten Chapters:

In the Introduction, the reader will find a short overview (from Chap. 2 to Chap. 10) of various subjects considered in this book.

In Chap. 1, we begin with some comments about the derivation of the basic, ‘exact,’ NSF model equations for real/viscous heat-conducting fluids

via the stress principle of Cauchy, Fourier's law, the conservation of total energy, and Gibbs's basic postulate for a homogeneous fluid. The dimensional form of the NSF equations is presented, and the various reduced nondimensional parameters in NSF equations and boundary conditions are discussed. Chapter 2 is devoted to some features and forms of the Navier, NS, and NSF model equations, and some simple (but fundamental) examples of viscous fluid flows are considered in Chap. 3. In Chaps. 4, 5, and 6, we discuss various implications of three main singular limits, the large viscosity limit, the vanishing viscosity limit, and the incompressible limit. For each of these cases, we give a theoretical account. Chapter 7 is a miscellany of various viscous fluid problems, and in the Introduction, the reader will find a detailed account of the subjects considered. Chapter 8 is devoted to basic tools for rigorous analysis of viscous (mainly Navier) equations and also to some information about recent rigorous mathematical results related to the existence and uniqueness of solutions of viscous model problems. In Chap. 9, some aspects of (hydrodynamic) stability theory are investigated, and in Chap. 10, we give a simple phenomenological presentation of the finite-dimensional dynamical system approach and routes to chaos/turbulence via strange attractors. This is followed by a 'collection' of examples of viscous fluid flow problems.

Chapters 8 to 10 cover a small part of the modern mathematical theory of viscous incompressible/compressible fluid flows.<sup>1</sup> Nevertheless, it was not our objective in these three chapters to give a detailed account of this mathematical theory, but rather from a fluid dynamicist's point of view, to provide an overview of recent results. In particular, Chap. 10 contains mostly arguments about current research but is essentially discursive.

The references cited (more than thousand) are listed at the end of the book before a detailed Index.

The choice of the 10 chapters mentioned and their order are, at least from our point of view, quite natural. The presentation of the material, the relative weight of the various arguments, and the general style reflect the tastes and knowledge of the author.

Naturally, the present book constitutes (with the companion book devoted to *Nonviscous Fluid Flows*) an advanced topic, rather than a classical course, on fluid dynamics. As for the subjects treated in these two volumes, I have been highly selective in my choice of topics. In many cases, the choice of subjects is based on my own interest and judgment. To that extent, the present text is a personal expression of my view ("à la Zeytounian"! ) of the theoretical viscous fluid dynamics.

The Contents enumerates the subjects in each chapter. This content of the book is ambitious but I hope that the reader will consider, after reading,

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<sup>1</sup> In the two volumes by Lions (*Mathematical Topics in Fluid Mechanics*, Volume 1; *Incompressible Models*, and Volume 2; *Compressible Models*, Oxford Lecture Series in Mathematics and its Applications, Clarendon Press, 1996 and 1998), the reader can find a rigorous mathematical theory for Navier and NS equations.

that I have (even partially) attained my objective – a modern presentation of some key problems in viscous fluid flows in terms of theoretical analysis, modeling and some applications.

Several colleagues have made many useful suggestions and criticisms during the preparation of this book for which I am grateful. However I accept final responsibility for remaining errors and omissions.

Finally, I thank: Dr. Christian Caron, Physics Editor, and the members of the Physical Editorial Department, where camera-ready manuscript in LaTeX was produced and reread by a native speaker.

Paris/Yport  
June 2003

*R.Kh. Zeytounian*

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