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Surfaces in 4-Space

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Scott dedicates his portion of the book to Zac and Leah.
Seiichi dedicates his portion to Naoko.
Masahico dedicates his portion to Shuko and Kaita.

Prologue

The earliest paper on knotted surfaces in 4-space of which we know is Artin's 1925 paper [Ar25] in which he described spinning. This paper initiated higher dimensional knot theory, and it indicated that such a theory is as rich as classical knot theory, since the fundamental groups of classical knots are realized as those of spun knots. Andrews and Curtis [AC59] proved that there exists an embedded sphere in the complement of the spun trefoil which represents a non-trivial element of the second homotopy group. This showed that the complement of a knotted sphere is in general not aspherical, in contrast to the classical case.

In 1941, Whitney studied non-orientable surfaces embedded in 4-space by means of what we now know as movies. In 1957, Fox and Milnor [FoxMil57*] formalized the motion picture method for studying knotted surfaces. This method allows constructing knotted surfaces that are not spun knots, and gives presentations of the fundamental group from such descriptions. The manuscript evolved into [Fox61a] and [FoxMil66]. In Fox's article "A Quick Trip through Knot Theory" [Fox61a] the motion picture method was popularized and several key examples were given. These examples indicate, by virtue of the structure of the fundamental group and the infinite cyclic cover, that knotted surfaces behave differently than classical knots. In 1961, Kinoshita [Kino61] proved that any polynomial $f(t)$ with $f(1) = \pm 1$ is realized as the Alexander polynomial of a knotted sphere, also showing the difference between knotted spheres and classical knots.

Yajima introduced the notion of simply knotted spheres, whose projections do not contain branch or triple points, and showed that they are ribbon knots [Yaji64]. Ribbon 2-knots became an important class of knots, for which geometric constructions and investigations of algebraic properties can be carried out with relative ease. Another important construction of knotted spheres, twist spinning, was introduced the following year by Zeeman [Ze65]. Zeeman's construction and its generalizations [Lith79] have been the main sources of non-ribbon 2-knots, yielding a plethora of examples that have features distinct from classical knots.

Thus the late 1950's and the early 1960's provided foundations in the development of the theory of knotted surfaces. During the following decades, ribbon, twist-spun, and generalizations together with variations of the motion picture method, remained the main sources of constructions. The fundamental groups and homological invariants of infinite cyclic covers were the main algebraic tools. After this period, the number of publications in the area increased drammatically.

Also during the last 40 years, a lot of excellent work was done on codimension 2 embedding in higher (than 4) dimensions. This work runs in parallel to and in conjunction with developments in higher dimensional differential topology, such as h-cobordism and surgery theories. The higher dimensional techniques include using the Whitney trick, for example, and thus these cannot be applied to surfaces in dimension 4. In this sense, knotted surfaces in 4-space are quite unique and mysterious.

In this book, we describe many of the developments in the case of surfaces embedded in 4-space. An in depth discussion of the developments in higher dimensions would take us far afield. Also, an encyclopedic treatise would be impossible. Thus we appologize here to our many colleagues whose contributions we had to omit.

Here is an overview of the book.

Chapter 1 discusses a variety of diagrammatic methods for depicting knotted surfaces. These include the movie or motion picture method, normal forms, marked vertex diagrams, broken surface diagrams, charts, and surface braids. We discuss methods to change these descriptions from one to another, and moves that relate isotopic surfaces with different diagrams. Proofs are omitted for the most part, but we point the reader to the source material.

Chapter 2 discusses many of the known methods of constructing knotted surfaces. The reader may prefer to read this in conjunction with Chapter 3 in which invariants are defined. The chapter opens with a discussion of spinning, twist-spinning, and deform spinning, in historical order. Ribbon surfaces are the subject of the next section. The connection to virtual knots is summarized, and some notions of equivalence of ribbon presentations are discussed. Connected sum in relation to the concordance group and uniqueness of factorizations are discussed. Satellite, cabling, surgery, and other constructions are reviewed. The notion of Seifert solids for knotted surfaces are given with a sketch of the Seifert algorithm. The chapter closes with a brief discussion of unknotting operations.

Chapter 3 reviews many of the invariants of knotted surfaces. The chapter opens with a description of the exterior of knotted surfaces, and handle decompositions thereof. In as much as is possible, we depict the handles or their cores near the various critical points. From the handle description of the exterior, a presentation of the fundamental group is given. The fundamental group has a Wirtinger presentation and we indicate how to compute this from the various diagrammatic and braid descriptions. The known characterizations of the possible fundamental groups are reviewed, and higher homotopy groups are discussed. Invariants from covering spaces are developed, including a definition of the Farber-Levine pairing and its generalization due to Kawauchi. The chapter closes with a discussion of analogues of linking numbers, and link homotopy invariants for singular knots.

Chapter 4 contains a detailed discussion of invariants that are derived from quandles. The chapter defines the fundamental quandle in the classical knot case and the knotted surface case, and exemplifies these for twist-spin knots. After a discussion on quandle colorings, we define quandle homology and cohomology theories and discuss these in analogy with that of groups. Indeed, 2-dimensional cocycles give rise to quandle extensions. The theory of quandle homology is a growing area. We summarize recent results of other authors in relation to this. To motivate the definition of the cocycle invariants, we sketch the definitions of the bracket model and the Dijkgraaf-Witten invariants. Then we present the quandle cocycle invariants for classical knots and for knotted surfaces. The chapter gives a geometric interpretation of quandle homology in terms of colored cobordism. It closes with applications of these invariants, and an epilogue.

Some of the work in this book appeared in [CarSai98a] and [Kama02], but much of it is new. The book [CarSai98a] focuses mainly on the diagrammatic theory of knotted surfaces. It indicates that a diagrammatic theory is not only possible but desirable from a categorical point of view. Meanwhile, [Kama02] contains a detailed description of surface braid theory. Both of these books emphasize authors' specific research interests and contributions, while the first three chapters of the current book emphasize historical developments and survey the subject. All of the material in Chapter 4 was developed after [CarSai98a] was written, and its genesis was contemporary with [Kama02]. We hope that the working geometric topologists will gain insight into the inner workings of knotted surfaces from all three of these works.

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