ИЗДАТЕЛЬСТВО · МИР ·

A Collection of Problems on the Equations of Mathematical Physics

Edited by V. S. Vladimirov

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Edited by Academician V. S. Vladimirov
Contributors: V. S. Vladimirov, V. P. Mikhailov,
A. A. Vasharin, Kh. Kh. Karimova, Yu. V. Sidorov,
M. I. Shabunin

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Preface

The extensive application of modern mathematical techniques to theoretical and mathematical physics requires a fresh approach to the course of equations of mathematical physics. This is especially true with regards to such a fundamental concept as the solution of a boundary value problem. The concept of a generalized solution considerably broadens the field of problems and enables solving from a unified position the most interesting problems that cannot be solved by applying classical methods. To this end two new courses have been written at the Department of Higher Mathematics at the Moscow Physics and Technology Institute, namely, "Equations of Mathematical Physics" by V.S. Vladimirov and "Partial Differential Equations" by V.P. Mikhailov (both books have been translated into English by Mir Publishers, the first in 1984 and the second in 1978).

The present collection of problems is based on these courses and amplifies them considerably. Besides the classical boundary value problems, we have included a large number of boundary value problems that have only generalized solutions. Solution of these requires using the methods and results of various branches of modern analysis. For this reason we have included problems in Lebesgue integration, problems involving function spaces (especially spaces of generalized differentiable functions) and generalized functions (with Fourier and Laplace transforms), and integral equations.

The book is aimed at undergraduate and graduate students in the physical sciences, engineering, and applied mathematics who have taken the typical "methods" course that includes vector analysis, elementary complex variables, and an introduction to Fourier series and boundary value problems. Asterisks denote the more difficult problems.

We would like to express our gratitude to all who helped with constructive comment to improve this book, our colleagues at the Department of Higher Mathematics at the Moscow Physics and Technology Institute, and especially T.F. Volkov, Yu.N. Drozhzhinov, A.F. Nikiforov, and V.I. Chekhlov.

V.S. Vladimirov
V.P. Mikhailov
A.A. Vasharin
Kh.Kh. Karimova
Yu.V. Sidorov
M.I. Shabunin

Contents

Symbols and Definitions

1. We denote a real n-dimensional Euclidean space by R^n and its points by $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, ξ and the like.

2. $dx = dx_1\, dx_2 \ldots dx_n$,

$$\int f(x)\, dx = \int\limits_{R^n} f(x_1, \ldots, x_n)\, dx_1 \ldots dx_n.$$

3. $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a multi-index, with the α_j nonnegative integers. We will also use the abbreviations

$$\alpha! = \alpha_1!\, \alpha_2! \ldots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

4. $(x, y) = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$

$$r = |x| = \sqrt{(x, x)} = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2}.$$

5. $U(x_0; R) = \{x: |x - x_0| < R\}$ is an *open ball* of radius R centered at point x_0, and $S(x_0; R) = \{x: |x - x_0| = R\}$ is a *sphere* of radius R centered at x_0; $U_R = U(0, R)$ and $S_R = S(0; R)$.

6. A set A will be said to be *lying strictly* in a region $G \subset R^n$ (this is denoted by $A \Subset G$) if it is bounded and $\overline{A} \subset G$.

7. A function $f(x)$ is said to be *locally integrable in a region* G if it is absolutely integrable in every subregion $G' \Subset G$. Functions that are locally integrable in R^n will be said to be simply *locally integrable.*

8. $$D^\alpha f(x) = \frac{\partial^{|\alpha|} f(x_1, x_2, \ldots, x_n)}{\partial x_1^{\alpha_1}\, \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

9. $C^p(G)$ is a class of functions f that are continuous together with the derivatives $D^\alpha f$, $|\alpha| \leqslant p$ $(0 \leqslant p < \infty)$ in the region $G \subset R^n$. The functions f of class $C^p(G)$ for which all the derivatives $D^\alpha f$, $|\alpha| \leqslant p$, allow continuous continuation into the closure $\overline{G}$ form the class $C^p(\overline{G})$; $C(G) = C^0(G)$, $C(\overline{G}) = C^0(\overline{G})$. We denote

the class of functions belonging to $C^p(G)$ for all p's by $C^\infty(G)$; the class $C^\infty(\bar{G})$ is defined in a similar manner.

10. The uniform convergence of a sequence of functions $\{f_k\}$ to a function f on a set A is denoted by

$$f_k(x) \overset{x \in A}{\Rightarrow} f(x), \quad k \to \infty$$

11. $A \cup B$ is the *union* of sets A and B, $A \cap B$ is the *intersection* of A and B, $A \setminus B$ is the *complement* of B with respect to A, and $A \times B$ is the *direct product* (or simply product) of A and B (the set of pairs (a, b) with $a \in A$ and $b \in B$).

12. The *support* of a continuous function f is denoted by supp f and is the closure of the set of all points x for which $f(x)$ is nonzero. If a function $f(x)$ that is measurable on a region G vanishes almost everywhere in $G \setminus G'$, where $G' \Subset G$, then it is *finite in* G; a function that is finite in R^n is said to be simply *finite*.

13. $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$ is Laplace's operator;

$\Box = \frac{\partial^2}{\partial t^2} - a^2\nabla^2$ is the wave operator; $\Box_1 = \Box$;

$\frac{\partial}{\partial t} - a^2\nabla^2$ is the heat conduction operator.

14. $\Gamma^+ = \{x, t: at > |x|\}$ is a future cone.

15. $\Phi(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\xi} e^{-z^2/2}\, dz$

16. $\omega_\varepsilon(x) = \begin{cases} C_\varepsilon e^{-\varepsilon^2/(\varepsilon^2 - |x|^2)} & |x| \leqslant \varepsilon, \\ 0, & |x| > \varepsilon, \end{cases}$

where $C_\varepsilon = \varepsilon^{-n}\varkappa \quad \frac{1}{\varkappa} = \int_0^1 e^{-1/(1-x^2)}\, dx$,

(ω_ε is the averaging kernel, or "cap").

17. $\mathbb{C}$ is the complex plane.

18. $\theta(x)$ is the Heaviside unit function:

$$\theta(x) = \begin{cases} 1 \text{ if } x \geqslant 0, \\ 0 \text{ if } x < 0. \end{cases}$$

19. $\sigma_n = \int_{S_1} ds = \frac{2\pi^{n/2}}{\Gamma(n/2)}$ is the surface area of the unit sphere S_1 in R^n.

20. In $C^p(\bar{G})$ the norm is

$$\| f \|_{C^p(\bar{G})} = \sum_{|\alpha| \leqslant p} \max_{x \in \bar{G}} |D^\alpha f(x)|$$

21. The totality of (measurable) functions $f(x)$ for which $|f|^p$ is integrable on G is denoted by $L_p(G)$. In $L_p(G)$ the norm is

$$\| f \|_{L_p(G)} = \left[\int_G |f|^p \, dx \right]^{1/p}. \quad 1 \leqslant p < \infty,$$

$$\| f \|_{L_\infty(G)} = \operatorname*{vrai\,sup}_{x \in G} |f(x)|, \quad p = \infty$$

The scalar product in $L_2(G)$ is introduced thus:

$$(f, g) = \int_G f \bar{g} \, dx, \quad f, g \in L_2(G).$$

22. Let $\rho(x)$ be a continuous positive-valued function in a region G. The totality of (measurable) functions $f(x)$ for which $\rho(x) \mid f(x) \mid^2$ is integrable on G is denoted by $L_{2,\rho}(G)$ and constitutes a Hilbert space with the scalar product

$$(f, g)_{L_{2,\rho}}(G) = \int_G \rho f \bar{g} \, dx.$$

23. Cylinder functions.

(a) Bessel functions:

$$J_\nu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+\nu+1)\,\Gamma(k+1)} \left(\frac{x}{2} \right)^{2k+\nu}, \quad -\infty < x < \infty;$$

(b) Neumann's Bessel functions:

$$N_\nu(x) = \frac{1}{\sin \pi\nu} [J_\nu(x) \cos \pi\nu - J_{-\nu}(x)], \quad \nu \neq n,$$

$$N_n(x) = \frac{1}{\pi} \left[\frac{\partial J_\nu(x)}{\partial \nu} - (-1)^n \frac{\partial J_{-\nu}(x)}{\partial \nu}, \quad \nu = n; \right.$$

(c) Hankel functions:

$$H_\nu^{(1)}(x) = J_\nu(x) + iN_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iN_\nu(x);$$

(d) modified Bessel and Hankel functions:

$$I_\nu(x) = e^{-\frac{\pi}{2}\nu i} J_\nu(ix), \quad K_\nu(x) = \frac{\pi i}{2} e^{\frac{\pi}{2}\nu i} H_\nu^{(1)}(ix).$$