Springer Monographs in Mathematics

Springer-Verlag Berlin Heidelberg GmbH

Helmut Koch

Galois Theory of p-Extensions

Springer

Helmut Koch
Humboldt-Universität Berlin
Institut für Mathematik
Unter den Linden 6
10117 Berlin
Gemany

Translator
Franz Lemmermeyer
California State University San Marcos
Department of Mathematics
333 South Twin Oaks Valley Rd.
92096-0001 San Marcos, CA
USA
e-mail: franzl@csusm.edu

Catalog-in-Publication Data applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Koch, Helmut: Galois theory of p-extensions / Helmut Koch.
ISBN 978-3-642-07817-0 ISBN 978-3-662-04967-9 (eBook)
DOI 10.1007/978-3-662-04967-9

Mathematics Subject Classification (2000): 11R34, 11-02

ISBN 978-3-642-07817-0

http://www.springer.de

Originally published by Springer-Verlag Berlin Heidelberg New York in 2002
Softcover reprint of the hardcover 1st edition 2002

Typeset in TEX by the Translator
Cover production: Erich Kirchner, Heidelberg

SPIN 10868507 44/3142db - 5 4 3 2 1 0 – Printed on acid-free paper

Foreword

This book deals with a quite young area of algebraic number theory: the algebraic theory of p-extensions, which was developed in the last 25 years and has now reached a degree of completion that makes a systematic presentation highly desirable.

This area of arithmetic deals with the theory of finite extensions of fields of arithmetic type. These are $\mathfrak{p}$-adic number fields, fields of formal power series with finite fields of constants, algebraic number fields and algebraic function fields of one variable with finite fields of constants. The main goal is obtaining information beyond what is provided by classical class field theory, which – as is well known – describes the extensions with commutative Galois group. The commutativity of the Galois group is very essential here. Class field theory is thereby closely connected with a wide circle of mathematical ideas ranging from the theory of radical extensions (nowadays called Kummer theory) to topological duality theorems, the theory of abelian and harmonic integrals and Picard varieties. The group-theoretical foundation of all these questions is the Pontryagin duality of abelian groups and their character groups. It is this part of mathematics that A. Weil has called "abelian mathematics".

As is well known, when Hilbert started building class field theory, he was led by the analogy between algebraic number fields and function fields, i.e., the field of meromorphic functions on compact Riemann surfaces. From this point of view, any "nonabelian" generalization of class field theory has to correspond to the investigation of the fundamental group of Riemann surfaces.

It has turned out that building a theory extending beyond the frame of class field theory is possible for extensions whose Galois group is nilpotent (or, what amounts to the same, a p-group). As in the study of the fundamental group in topology, it emerged that the groups of interest here have a natural presentation in terms of generators and relations, which clarifies essential parts of their structure. These investigations are contained in this book. Although the theory presented here is far from dealing with the general case of finite groups, it leads to the solution of many number-theoretic problems. For example, the theory says a lot about the structure of *all* extensions of a $\mathfrak{p}$-adic number field or of the field of formal power series over a finite field of constants. This theory also leads to the solution of the class field tower problem and of the problem connected with it, the growth of minimal discriminants of algebraic number fields.

At the same time I would like to stress that we are dealing here with a theory that is not yet complete. On the contrary, several extremely interesting problems remain unsolved. Examples of such questions are the following: a concrete description of the Galois group of the algebraic closure of a $\mathfrak{p}$-adic number field (e.g. including the ramification groups), the existence of extension fields of the rational number field with given (nonsolvable) Galois group, and the integrality of L-series for Artin characters different from the trivial character.

I am convinced that this book is of interest to a large circle of mathematicians. On the one hand, it is accessible to nonexperts and leads the reader quickly into a new area of investigation that contains a wealth of problems awaiting solution. On the other hand, the book contains almost all the basic material pertaining to this area as well as many new results by the author that will be of interest to the experts. The author's papers are among the most interesting results in this direction and have contributed essentially to the reputation that this area of mathematics has nowadays.

I hope that this book by H. Koch will stimulate the further development of this direction in algebraic number theory.

Moscow, December 1969 I.R. Shafarevich

Preface for the English Translation

There are a number of reasons why an English translation of this book appears more than 30 years after its publication.

1. It is very hard to find a copy of this book even from used book stores. On the other hand, it is still in demand since the subject has not been included in textbooks.

2. This book demonstrates that the cohomology of groups is very useful for studying Galois theory of number fields; at the same time, it offers a down to earth introduction to the cohomological method. Thus it may be read in parallel to Serre's classical lectures *Cohomologie Galoisienne* first published in the Springer Lecture Notes Series in 1964, and whose English translation appeared in 2000.

3. Another book published in 2000 is the encyclopedic [NSW] by J. Neukirch, A. Schmidt and K. Wingberg, where the results presented in this book are discussed only marginally. For example, the authors refer to the German original of this book for the improved version of the Golod-Shafarevich inequality. On the other hand, [NSW] contains new results on the Galois theory of p-extensions, in particular in connection with maximal p-extensions unramified outside of a set S of primes containing the primes dividing p. The present book, on the other hand, concentrates on extensions in which primes dividing p may be unramified.

Berlin, autumn 2001 H. Koch

Note by the Translator

It is my pleasure to thank Linda Holt for her comments, Farshid Hajir for suggesting the translation in the first place and for his contribution to the postscript, Wayne Aitken for corrections and explanations, Richard Hill, Hendrik Lenstra, Akito Nomura, Peter Pleasant and an unknown referee for lists of typos and errors, and Helmut Koch for his help in preparing this translation.

San Marcos, spring 2002 F. Lemmermeyer

Preface of the German Edition

The main objective of this book is a consistent presentation of results by I.R. Shafarevich, A. Fröhlich, A. Brumer and the author on the Galois theory of p-extensions on the basis of Galois cohomology. In order to make these results accessible to a wider circle of mathematicians with algebraic interests, prerequisites are restricted to knowledge of basic facts from algebra, group theory, and algebraic number theory as they are readily available in standard textbooks.

The first seven chapters of this book deal with cohomology of profinite groups, in particular of pro-p-groups. It goes without saying that the first chapter of Serre's Cohomologie Galoisienne served as a blueprint. Some chapters of this part may be seen as a commentary on Serre's lectures. The rest of the book deals with field theory. The theorems of class field theory that we need are stated explicitly and may be accepted axiomatically by the readers. There exist, by the way, at least two easily accessible presentations of class field theory in the style and extent in which we will need it, namely J.W.S. Cassels and A. Fröhlich [8] as well as J. Neukirch [46].

Some of the results of this book I found during a one-year stay in 1967/68 as a researcher at the Steklov Institute for Mathematics of the Academy of Science of the USSR. I would like to express my heartfelt gratitude to the Steklov Institute and in particular to Prof. I.R. Shafarevich for the invitation. I.R. Shafarevich's influence on this book is much larger, though: it goes back to my first stay in Moscow in 1960/61 as a student, which was followed by numerous inspiring and stimulating conversations.

I would like to thank the editors, in particular Prof. Dr. H. Reichardt, for including this book into the series "Mathematische Monographien". Moreover I thank Prof. Dr. H. Reichardt and the members of the number theory research group of the Institute for Pure Mathematics at the German Academy of Science in Berlin, namely Dr. O. Neumann, W. Thor, H. Pieper, and W. Zink, who have read parts of the manuscript and suggested improvements and corrections. I thank the VEB Deutscher Verlag der Wissenschaften, in particular the copy editors L. Boll and Miss E. Arndt, for their exemplary and expert work on the manuscript, which has contributed essentially to the success of this publication.

Berlin, autumn 1969 H. Koch

Contents