

Universitext

**Springer-Verlag Berlin Heidelberg GmbH**

Jürgen Jost

# Compact Riemann Surfaces

An Introduction  
to Contemporary Mathematics

Second Edition



Springer

Jürgen Jost  
Max Planck Institute for Mathematics in the Sciences  
Inselstr. 22–26  
04103 Leipzig  
Germany

Library of Congress Cataloging-in Publication Data applied for

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Jost, Jürgen: Compact Riemann surfaces: an introduction to contemporary  
mathematics / Jürgen Jost. – 2. ed..

Einheitssacht.: Riemannsche Flächen <engl.>

ISBN 978-3-540-43299-9

ISBN 978-3-662-04745-3 (eBook)

DOI 10.1007/978-3-662-04745-3

Mathematics Subject Classification (2000): 30F10, 30F45, 30F60, 58E20, 14H55

ISBN 978-3-540-43299-9

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

<http://www.springer.de>

© Springer-Verlag Berlin Heidelberg 2002

Originally published by Springer-Verlag Berlin Heidelberg New York in 2002

The use of general descriptive names, registered names, trademarks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

Typesetting: Camera-ready copy produced by the author using a Springer T<sub>E</sub>X macro package

Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 10866725 41/3142/db – 5 4 3 2 1 0

**Dedicated to my parents**

# Preface

The present book started from a set of lecture notes for a course taught to students at an intermediate level in the German system (roughly corresponding to the beginning graduate student level in the US) in the winter term 86/87 in Bochum. The original manuscript has been thoroughly reworked several times although its essential aim has not been changed.

Traditionally, many graduate courses in mathematics, and in particular those on Riemann surface theory, develop their subject in a most systematic, coherent, and elegant manner from a single point of view and perspective with great methodological purity. My aim was instead to exhibit the connections of Riemann surfaces with other areas of mathematics, in particular (two-dimensional) differential geometry, algebraic topology, algebraic geometry, the calculus of variations and (linear and nonlinear) elliptic partial differential equations. I consider Riemann surfaces as an ideal meeting ground for analysis, geometry, and algebra and as ideally suited for displaying the unity of mathematics. Therefore, they are perfect for introducing intermediate students to advanced mathematics. A student who has understood the material presented in this book knows the fundamental concepts of algebraic topology (fundamental group, homology and cohomology), the most important notions and results of (two-dimensional) Riemannian geometry (metric, curvature, geodesic lines, Gauss-Bonnet theorem), the regularity theory for elliptic partial differential equations including the relevant concepts of functional analysis (Hilbert- and Banach spaces and in particular Sobolev spaces), the basic principles of the calculus of variations and many important ideas and results from algebraic geometry (divisors, Riemann-Roch theorem, projective spaces, algebraic curves, valuations, and many others). Also, she or he has seen the meaning and the power of all these concepts, methods, and ideas at the interesting and nontrivial example of Riemann surfaces.

There are three fundamental theorems in Riemann surface theory, namely the uniformization theorem that is concerned with the function theoretic aspects, Teichmüller's theorem that describes the various conformal structures on a given topological surface and for that purpose needs methods from real analysis, and the Riemann-Roch theorem that is basic for the algebraic geometric theory of compact Riemann surfaces. Among those, the Riemann-Roch theorem is the oldest one as it was rigorously demonstrated and successfully

applied already by the middle of the last century. The uniformization theorem was stated by Riemann as well, but complete proofs were only found later by Poincaré and Koebe. Riemann himself had used the so-called Dirichlet principle for the demonstration of that result which, however, did not withstand Weierstrass' penetrating criticism and which could be validated only much later by Hilbert. In any case, it seems that the algebraic geometry of Riemann surfaces had a better start than the analysis which succeeded only in our century in developing general methods. Teichmüller's theorem finally is the youngest one among these three. Although the topological result was already known to Fricke and Klein early this century, it was Teichmüller who in the thirties worked out the fundamental relation between the space that nowadays bears his name and holomorphic quadratic differentials. Teichmüller himself was stimulated by earlier work of Grötzsch. Complete proofs of the results claimed by Teichmüller were only provided by Ahlfors and Bers in the fifties and sixties.

In the present book, all three fundamental theorems are demonstrated (we treat only compact Riemann surfaces; while the Riemann-Roch and Teichmüller theorems are naturally concerned with compact surfaces, for the uniformization theorem this means that we restrict to an easier version, however). For Riemann-Roch, we give an essentially classical proof. Teichmüller's theorem is usually derived with the help of quasiconformal mappings. Here, we shall present a different approach using so-called harmonic maps instead. This method will also be used for the uniformization theorem. While quasiconformal maps are defined by a pointwise condition, harmonic maps are obtained from a global variational principle. Therefore, the analytic properties of harmonic maps are better controlled than those of quasiconformal maps. For both classes of maps, one needs the regularity theory of elliptic partial differential equations, although harmonic maps are perhaps a little easier to treat because they do not require the Calderon-Zygmund theorem. What is more important, however, is that harmonic map theory is of great use in other areas of geometry and analysis. Harmonic mappings are critical points for one of the simplest nonlinear geometrically defined variational problems. Such nonlinear methods have led to enormous progress and far-reaching new developments in geometry. (Let us only mention Yau's solution of the Calabi conjecture that is concerned with differential equations of Monge-Ampère type, with its many applications in algebraic geometry and complex analysis, the many applications that harmonic maps have found for Kähler manifolds and symmetric spaces, and the breakthroughs of Donaldson in four-dimensional differential topology that were made possible by using Yang-Mills equations, and most recently, the Seiberg-Witten equations.) The present book therefore is also meant to be an introduction to nonlinear analysis in geometry, by showing the power of this approach for an important and interesting example, and by developing the necessary tools. This constitutes the main new aspect of the present book.

As already indicated, and as is clear from the title anyway, we only treat compact Riemann surfaces. Although there exists an interesting and rich theory of noncompact (open) Riemann surfaces as well, for mathematics as a whole, the theory of compact Riemann surfaces is considerably more important and more central.

Let us now describe the contents of the present book more systematically.

The first chapter develops some topological material, in particular fundamental groups and coverings, that will be needed in the second chapter.

The second chapter is mainly concerned with those Riemann surfaces that are quotients of the Poincaré upper half plane (or, equivalently, the unit disk) and that are thus equipped with a hyperbolic metric. We develop the foundations of two-dimensional Riemannian geometry. We shall see the meaning of curvature, and, in particular, we shall discuss the Gauss-Bonnet theorem in detail, including the Riemann-Hurwitz formula as an application. We also construct suitable fundamental polygons that carry topological information. We also treat the Schwarz lemma of Ahlfors and its applications, like the Picard theorem, thus illustrating the importance of negatively curved metrics, and discussing the concept of hyperbolic geometry in the sense of Kobayashi. Finally, we discuss conformal structures on tori; apart from its intrinsic interest, this shall serve as a preparation for the construction of Teichmüller spaces in the fourth chapter. In any case, one of the main purposes of the second chapter is to develop the geometric intuition for compact Riemann surfaces.

The third chapter is of a more analytic nature. We briefly discuss Banach and Hilbert space and then introduce the Sobolev space of square integrable functions with square integrable weak derivatives, i.e. with finite Dirichlet integral. This is the proper framework for Dirichlet's principle, i.e. for obtaining harmonic functions by minimizing Dirichlet's integral. One needs to show differentiability properties of such minimizers, in order to fully justify Dirichlet's principle. As an introduction to regularity theory for elliptic partial differential equations, we first derive Weyl's lemma, i.e. the smoothness of weakly harmonic functions. For later purposes, we also need to develop more general results, namely the regularity theory of Korn, Lichtenstein, and Schauder that works in the  $C^{k,\alpha}$  Hölder spaces. We shall then be prepared to treat harmonic maps, our central tool for Teichmüller theory and the uniformization theorem in an entirely elementary manner, we first prove the existence of energy minimizing maps between hyperbolic Riemann surfaces; the previously developed regularity theory will then be applied to show smoothness of such minimizers. Thus, we have found harmonic maps. Actually, the energy integral is the natural generalization of Dirichlet's integral for maps into a manifold - hence also the name "harmonic maps". We shall then show that under appropriate assumptions, harmonic maps are unique and diffeomorphisms. Incidentally, Hurwitz' theorem about the finiteness of the number of automorphisms of a compact Riemann surface of genus  $p = 1$



is a direct consequence of the uniqueness of harmonic maps in that case.

The fourth chapter is concerned with Teichmüller theory. Our starting point is the observation that a harmonic map between Riemann surfaces naturally induces some holomorphic object, a so-called holomorphic quadratic differential on the domain. We investigate how this differential changes if we vary the target while keeping the domain fixed. As a consequence, we obtain Teichmüller's theorem that Teichmüller space is diffeomorphic to the space of holomorphic quadratic differentials on a fixed Riemann surface of the given genus. This bijection between marked conformal structures and holomorphic quadratic differentials is different from the one discovered by Teichmüller and formulated in terms of extremal quasiconformal maps. We also introduce Fenchel-Nielsen coordinates on Teichmüller space as an alternative approach for the topological structure of Teichmüller space. Finally, using similar harmonic map techniques as in the proof of Teichmüller's theorem, we also demonstrate the uniformization for compact Riemann surfaces; here, the case of surfaces of genus 0 requires a somewhat more involved construction than the remaining ones.

The last chapter finally treats the algebraic geometry of Riemann surfaces, historically the oldest aspect of the subject. Some of the central results had already been derived by Abel and Jacobi even before Riemann introduced the concept of a Riemann surface. We first introduce homology and cohomology groups of compact Riemann surfaces and building upon that, harmonic, holomorphic, and meromorphic differential forms. We then introduce divisors and derive the Riemann-Roch theorem. As an application, we compute the dimensions of the space of holomorphic quadratic differentials on a given Riemann surfaces, and consequently also the dimension of the corresponding Teichmüller space that was the subject of the fourth chapter. We also obtain projective embeddings of compact Riemann surfaces. We then study the field of meromorphic functions on a compact Riemann surface and realize such surfaces as algebraic curves. We also discuss the connection with the algebraic concept of a field with a valuation. We then prove Abel's famous theorem on elliptic functions and their generalizations or - in different terminology - on linearly equivalent divisors, as well as the Jacobi inversion theorem. The final section discusses the preceding results for the beautiful example of elliptic curves.

Often, we shall use the terminology of modern algebraic geometry instead of the classical one; however, the notions of sheaf theory have not been used.

The prerequisites are mostly of an elementary nature; clearly, for understanding and appreciating the contents of the present book, some previous exposure to mathematical reasoning is required. We shall need some fundamental results from real analysis, including Lebesgue integration theory and the  $L^p$ -spaces, which can be found in my textbook "Postmodern Analysis" (see the bibliography). We shall also obviously require some background from complex analysis (function theory), but definitely not going beyond what is con-

tained in Ahlfors' "Complex Analysis". In particular, we assume knowledge of the following topics: holomorphic functions and their elementary properties, linear transformations (in our book called "Möbius transformations"), the residue theorem, the Arzelà-Ascoli theorem. At some isolated places, we use some results about doubly periodic meromorphic functions, and in Sec. 5.10 also some properties of the Weierstrass  $\mathcal{P}$ -function.

Finally, in Sec. 5.8, for purposes of motivation only, from the last chapter of Ahlfors, we recall the construction of a Riemann surface of an algebraic function as a branched cover of the two-sphere.

In Sec. 5.1, we require some basic results about analysis on manifolds, like the Stokes and Frobenius theorems.

For writing the present book, I have used rather diverse sources as detailed at the end. (All sources, as well as several additional references for further study, are compiled in the bibliography.) In particular, I have attributed the more recent theorems derived in the text to their original authors in that section, instead of the main text. Historical references to the older literature are sparse since so far, I did not enjoy the leisure required to check this carefully.

At the end of most sections, some exercises are given. The more demanding ones are marked by an asterisk.

I thank R. R. Simha for his competent translation of my original German manuscript into English, for his several useful suggestions for improvements, and in particular for his enthusiasm and good will in spite of several mishaps. Tilmann Wurzbacher and Wolfgang Medding kindly supplied many useful and detailed corrections and suggestions for my manuscript. Several corrections were also provided by Marianna Goldcheid and Jochen Lohkamp. The book benefited extremely from the thorough and penetrating criticism and the manifold suggestions that were offered by Jürgen Büser.

Finally, I am grateful to Isolde Gottschlich, Erol Karakas, Michael Knebel, and Harald Wenk for typing and retyping various versions of my manuscript.

## Preface to the 2<sup>nd</sup> edition

The subject of Riemann surfaces is as lively and important as ever. In particular, Riemann surfaces are the basic geometric objects of string theory, the physical theory aiming at a unification of all known physical forces. String theory starts with a one-dimensional object, a string, and as such a string moves in space-time, it sweeps out a surface. What is relevant about this surface is its conformal structure, and so we are naturally led to the concept of a Riemann surface. In fact, much of string theory can be developed on the basis of the results, constructions, and methods presented in this book, and I have explored this approach to string theory in my recent monograph “Bosonic strings: A mathematical treatment”, AMS and International Press, 2001.

For this new edition, I have streamlined the presentation somewhat and corrected some misprints and minor inaccuracies.

I thank Antje Vandenberg for help with the  $\text{\TeX}$  work.

Leipzig, February 2002

*Jürgen Jost*

# Table of Contents

<b>Preface</b> .....	VII
<b>1. Topological Foundations</b> .....	1
1.1 Manifolds and Differentiable Manifolds .....	1
1.2 Homotopy of Maps. The Fundamental Group .....	3
1.3 Coverings .....	7
1.4 Global Continuation of Functions on Simply-Connected Manifolds .....	16
<b>2. Differential Geometry of Riemann Surfaces</b> .....	19
2.1 The Concept of a Riemann Surface .....	19
2.2 Some Simple Properties of Riemann Surfaces .....	22
2.3 Metrics on Riemann Surfaces .....	23
2.3.A Triangulations of Compact Riemann Surfaces .....	34
2.4 Discrete Groups of Hyperbolic Isometries. Fundamental Polygons. Some Basic Concepts of Surface Topology and Geometry .....	41
2.4.A The Topological Classification of Compact Riemann Surfaces .....	55
2.5 The Theorems of Gauss-Bonnet and Riemann-Hurwitz ....	59
2.6 A General Schwarz Lemma .....	66
2.7 Conformal Structures on Tori .....	74
<b>3. Harmonic Maps</b> .....	81
3.1 Review: Banach and Hilbert Spaces. The Hilbert Space $L^2$ .	81
3.2 The Sobolev Space $W^{1,2} = H^{1,2}$ .....	93
3.3 The Dirichlet Principle. Weak Solutions of the Poisson Equation .....	102
3.4 Harmonic and Subharmonic Functions .....	106
3.5 The $C^\alpha$ -Regularity Theory .....	113
3.6 Maps Between Surfaces. The Energy Integral. Definition and Simple Properties of Harmonic Maps .....	120
3.7 Existence of Harmonic Maps .....	126
3.8 Regularity of Harmonic Maps .....	135

3.9	Uniqueness of Harmonic Maps .....	140
3.10	Harmonic Diffeomorphisms .....	145
3.11	Metrics and Conformal Structures .....	154
<b>4.</b>	<b>Teichmüller Spaces .....</b>	<b>161</b>
4.1	The Basic Definitions .....	161
4.2	Harmonic Maps, Conformal Structures and Holomorphic Quadratic Differentials. Teichmüller's Theorem .....	164
4.3	Fenchel-Nielsen Coordinates. An Alternative Approach to the Topology of Teichmüller Space .....	174
4.4	Uniformization of Compact Riemann Surfaces .....	184
<b>5.</b>	<b>Geometric Structures on Riemann Surfaces .....</b>	<b>189</b>
5.1	Preliminaries: Cohomology and Homology Groups .....	189
5.2	Harmonic and Holomorphic Differential Forms on Riemann Surfaces .....	197
5.3	The Periods of Holomorphic and Meromorphic Differential Forms .....	205
5.4	Divisors. The Riemann-Roch Theorem .....	211
5.5	Holomorphic 1-Forms and Metrics on Compact Riemann Surfaces .....	222
5.6	Divisors and Line Bundles .....	224
5.7	Projective Embeddings .....	234
5.8	Algebraic Curves .....	242
5.9	Abel's Theorem and the Jacobi Inversion Theorem .....	254
5.10	Elliptic Curves .....	262
	<b>Sources and References .....</b>	<b>269</b>
	<b>Bibliography .....</b>	<b>271</b>
	<b>Index of Notation .....</b>	<b>273</b>
	<b>Index .....</b>	<b>275</b>