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Function Spaces and Potential Theory

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Preface

Function spaces, especially those spaces that have become known as Sobolev spaces, and their natural extensions, are now a central concept in analysis. In particular, they play a decisive role in the modern theory of partial differential equations (PDE).

Potential theory, which grew out of the theory of the electrostatic or gravitational potential, the Laplace equation, the Dirichlet problem, etc., had a fundamental role in the development of functional analysis and the theory of Hilbert space. Later, potential theory was strongly influenced by functional analysis. More recently, ideas from potential theory have enriched the theory of those more general function spaces that appear naturally in the study of nonlinear partial differential equations. This book is motivated by the latter development.

The connection between potential theory and the theory of Hilbert spaces can be traced back to C. F. Gauss [181], who proved (with modern rigor supplied almost a century later by O. Frostman [158]) the existence of equilibrium potentials by minimizing a quadratic integral, the energy. This theme is pervasive in the work of such mathematicians as D. Hilbert, Ch.-J. de La Vallée Poussin, M. Riesz, O. Frostman, A. Beurling, and the connection was made particularly clear in the work of H. Cartan [97] in the 1940's. In the thesis of J. Deny [119], and in the subsequent work of J. Deny and J. L. Lions [122] in the early 1950's, this point of view, blended with the L. Schwartz theory of distributions, led to a new understanding of the function spaces of Hilbert type.

According to the classical Dirichlet principle, one obtains the solution of Dirichlet's problem for the Laplace equation in a domain Ω by minimizing the Dirichlet integral, $\int_\Omega |\nabla u(x)|^2\,dx$, over a certain class of functions taking given values on the boundary $\partial\Omega$. The natural explanation is that solutions to the Laplace equation describe an equilibrium state, a state attained when the energy carried by the system is at a minimum.

Also, the classical electrostatic capacity (or capacitance, as it is known to physicists) of a compact set $K \subset \mathbf{R}^3$ differs only by a constant factor from $C(K) = \inf \int_\Omega |\nabla u(x)|^2\,dx$, where the infimum is taken over all smooth compactly supported functions u, such that $u \geq 1$ on K.

As emphasized by H. Cartan, taking these infima of the Dirichlet integral is equivalent to taking projections in a Hilbert space normed by the square root of the Dirichlet integral.

A slight modification of the norm defined by the Dirichlet integral leads to the simplest of the function spaces treated in this work, the Sobolev space denoted by H^1 or $W^{1,2}$. It is normed by the square root of $\|u\|_{H^1}^2 = \int_{\mathbf{R}^N} \left(|u|^2 + |\nabla u|^2\right) dx$. This Hilbert space is an indispensable tool in the study of linear PDE of the second order. The higher Sobolev spaces, H^m or $W^{m,2}$, whose definition involves derivatives of order up to m, are similarly related to linear PDE of order $2m$.

For the study of nonlinear equations it is advantageous to introduce Sobolev spaces $W^{m,p}$, modeled on L^p. The methods of classical potential theory based on a Hilbert space approach are insufficient to investigate analogous questions in these spaces at the same depth as those modeled on L^2, and this has led to the creation of a new, nonlinear potential theory. This potential theory can be viewed as a natural extension of the classical, linear theory, and when specialized to $p = 2$, it includes a surprisingly large part of the linear theory as a special case.

The present book is devoted to this interplay of potential theory and function spaces, with the purpose of studying the properties of functions belonging to Sobolev spaces, or to some of their natural extensions, such as Bessel potential spaces, Besov spaces, and Lizorkin–Triebel spaces.

Although there are earlier roots, one can date the birth of this theory to the work of V. G. Maz′ya and J. Serrin in the early 1960's. The theory took a new turn in the writings of several authors in the years around 1970: B. Fuglede, N. G. Meyers, Yu. G. Reshetnyak, V. P. Havin, and V. G. Maz′ya. Over the last decades it has continued to develop, and it has found numerous applications. It has greatly clarified the properties of elements of Sobolev spaces and their generalizations, and many problems have been given definitive answers in terms of this theory. Some of these originate with the book of L. Carleson [92], and part of the original motivation for writing this book was to make these results more easily accessible. By now the theory has reached a level of maturity and beauty that makes the time ripe for a book on the subject.

The germ of the book lies in lectures given by the authors — by the Swedish author at Indiana University, Bloomington, Indiana, during the academic year 1978–79, and by the American author at the University of Umeå, Sweden, during the months of March and April 1981. The idea of writing a joint book goes back almost to the same time. The first synopsis was in fact written during the AMS Summer Research Institute at Berkeley in the summer of 1983, although the writing did not start in earnest until some six years later.

The reader we have in mind should have a good graduate course in real analysis, but is not required to know anything about capacities and potentials. We often refer to the well-known books by W. Rudin [367, 368], and E. M. Stein [389] for basic facts needed herein.

We describe very briefly the contents of the book. More information is found in the introduction of each chapter.

Chapter 1 gives some background, and can be consulted as needed. Chapter 2 contains some of the central material of the book, especially in Sections 2.2, 2.3, and 2.5. This includes definitions of (α, p)-capacities and the associated nonlinear potentials. The contents are described in more detail in Section 2.1.

Chapter 3, which is largely independent of the theory of Chapter 2, is mainly devoted to a number of important inequalities. In Chapter 4 we show that the general theory of Chapter 2, which was first developed with Sobolev spaces in mind, can also be applied to spaces of Besov and Lizorkin–Triebel type, and that their representations by "smooth atoms" are well suited for this purpose. Since we do not want to assume any previous knowledge of these spaces, we present their theory from the beginning.

Chapter 5 is mainly devoted to comparisons of (α, p)-capacities with other, better known set functions (Hausdorff measures). The results are sharp, and extend estimates which are well known in the case $p = 2$. In Chapter 6 we apply the theory to a close study of local continuity properties of functions in Sobolev (and more general) spaces, including a non-trivial extension of the classical theory of thin sets. Chapters 7 and 8 are devoted to trace and imbedding theorems, and to inequalities of Poincaré type, respectively.

Chapters 6–8 show that many questions in analysis can be given final answers in terms of (α, p)-capacities, capacities which are shown in Chapter 5 to be very well understood.

Chapters 9 and 10 have a character which differs somewhat from the preceding chapters. They are mainly devoted to the proof of a theorem which describes the kernel of a trace operator on arbitrary sets, but they use different methods. In Chapter 9 the methods are mainly potential theoretical, and depend on the results of Chapters 6 and 8. In Chapter 10 a more general result due to Yu. V. Netrusov is proved using the powerful methods presented in Chapter 4.

In the final Chapter 11, most of which can be read immediately after Chapter 6, we give complete solutions in terms of capacities to some approximation problems for holomorphic and harmonic functions.

Netrusov communicated the results presented in Chapter 10 to the authors in December 1991, which necessitated some changes in the original plans for the book, and he presented detailed proofs during visits to Linköping in 1992 and 1993. The authors are most grateful to him for permitting the inclusion of these results and their proofs here before their full publication elsewhere, and for his most valuable help in preparing the presentation.

In each chapter most references to the literature are collected in "Notes" at the end of the chapter. These notes are sometimes quite detailed, but there is no uniformity in the depth to which the history of different ideas is traced. The discussion reflects the authors' interests and knowledge, and is not based on systematic historical research. Concerning the difficulties involved in writing such historical remarks we refer to J. L. Doob [125], p. 793.

The same apologies apply to the "Further Results" sections, where we give a number of results that complement the main body of the text.

The bibliography, although quite extensive, is limited to works mentioned in the text, and does not pretend to completeness.

There is not much overlap between this book and other books on function spaces. In fact, most of the contents have not previously been presented in book form. Also, there are many parts of the theory that are not treated by us. An important omission is the Hardy spaces H^p for $p \leq 1$, and spaces derived from them, and also the related theory of interpolation spaces. These subjects are treated in many places, e.g. in several books by H. Triebel [404, 405, 406]. A second omission is the space BV of functions of bounded variation and related spaces, treated e.g. in the book by W. P. Ziemer [438]. We also leave aside the situation in spaces defined on domains with irregular boundaries. This is an area with many open problems, and we refer to the book by V. G. Maz'ya [308] for more information on this and many other subjects.

Finally, we largely omit the nonlinear potential theory which is special to quasilinear second order PDE. A natural generalization of the Laplace equation is obtained by minimizing the integral $\int_\Omega |\nabla u(x)|^p\, dx$ over functions taking given boundary values. This leads to the so called p-Laplace equation, $\Delta_p u = 0$, with the p-Laplacian Δ_p defined by $\Delta_p u = \operatorname{div}(\nabla u|\nabla u|^{p-2})$. However, there is no potential representation of solutions to the equation $\Delta_p u = f$ for $p \neq 2$ which corresponds to the Newtonian potential in the case of the Laplacian. Nevertheless, concepts like superharmonic functions, harmonic measure, and the Perron method for solving the Dirichlet problem have been successfully extended to this setting. This theory is (at least for the time being) limited to second order equations and inequalities, and consequently it is concerned with Sobolev spaces (with and without weight) of order one. But it should be noted that, although the methods are quite different, there are many parallels and similarities with the theory presented in this book, especially in such key places as the theory of capacity and the theory of thin sets. We refer the reader to the recent book by J. Heinonen, T. Kilpeläinen, and O. Martio [221] for an exposition.

The authors owe a large dept of gratitude to Yu. V. Netrusov. In addition to his contributions described above, he has read the entire manuscript carefully, and suggested a number of significant improvements of the presentation. The more important ones of these are acknowledged in the text. We are also grateful to V. G. Maz'ya, Linköping, for many enlightening comments.

The fact that both authors have worked in this area for many years does not diminish their feelings of gratitude to those who taught them mathematics, and above all to L. Carleson, and N. G. Meyers. They have influenced this book by their teaching and example, and directly by their work, some of which was mentioned above.

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