Universitext

Bernt Øksendal

Stochastic Differential Equations

An Introduction with Applications

Second Edition

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The front cover shows the graph of a sample path of a geometric Brownian motion, i.e. the solution X_t of the (1-dimensional) stochastic differential equation

$$\frac{dX_t}{dt} = (r + \alpha \cdot W_t)X_t, \; X_0 = 1,$$

where W_t is white noise. This process is often used to model "exponential growth under uncertainty". See Chapters 5, 10 and 11.
The figure is a computer simulation for the case $\alpha = 0.4$, $r = 1.08$. The mean value of X_t, $\exp(rt)$, is also drawn, $0 \leq t \leq 3$. It was produced by courtesy of Tore Jonassen, University of Oslo.

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To my family
Eva, Elise, Anders, and Karina

Preface

In the second edition I have split the chapter on diffusion processes in two, the new Chapters VII and VIII:
Chapter VII treats only those basic properties of diffusions that are needed for the applications in the last 3 chapters. The readers that are anxious to get to the applications as soon as possible can therefore jump directly from Chapter VII to Chapters IX, X and XI.

In Chapter VIII other important properties of diffusions are discussed. While not strictly necessary for the rest of the book, these properties are central in today's theory of stochastic analysis and crucial for many other applications.

Hopefully this change will make the book more flexible for the different purposes. I have also made an effort to improve the presentation at some points and I have corrected the misprints and errors that I knew about, hopefully without introducing new ones. I am grateful for the responses that I have received on the book and in particular I wish to thank Henrik Martens for his helpful comments.

Tove Lieberg has impressed me with her unique combination of typing accuracy and speed. I wish to thank her for her help and patience, together with Dina Haraldsson and Tone Rasmussen who sometimes assisted on the typing.

Oslo, August 1989 Bernt Øksendal

Preface to the First Edition

These notes are based on a postgraduate course I gave on stochastic differential equations at Edinburgh University in the spring 1982. No previous knowledge about the subject was assumed, but the presentation is based on some background in measure theory.

There are several reasons why one should learn more about stochastic differential equations: They have a wide range of applications outside mathematics, there are many fruitful connections to other mathematical disciplines and the subject has a rapidly developing life of its own as a fascinating research field with many interesting unanswered questions.

Unfortunately most of the literature about stochastic differential equations seems to place so much emphasis on rigor and completeness that is scares many nonexperts away. These notes are an attempt to approach the subject from the nonexpert point of view: Not knowing anything (except rumours, maybe) about a subject to start with, what would I like to know first of all? My answer would be:

1) In what situations does the subject arise?
2) What are its essential features?
3) What are the applications and the connections to other fields?

I would not be so interested in the proof of the most general case, but rather in an easier proof of a special case, which may give just as much of the basic idea in the argument. And I would be willing to believe some basic results without proof (at first stage, anyway) in order to have time for some more basic applications.

These notes reflect this point of view. Such an approach enables us to reach the highlights of the theory quicker and easier. Thus it is hoped that notes may contribute to fill a gap in the existing literature. The course is meant to be an appetizer. If it succeeds in awaking further interest, the reader will have a large selection of excellent literature available for the study of the whole story. Some of this literature is listed at the back.

In the introduction we state 6 problems where stochastic differential equations play an essential role in the solution. In Chapter II we introduce the basic mathematical notions needed for the mathematical model of some of these problems, leading to the concept of Ito integrals in Chapter III. In Chapter IV we develop the stochastic calculus (the Ito formula) and in Chapter V we use this to solve some stochastic differential equations, including the first two problems in the introduction. In Chapter VI we present a solution of the linear filtering problem (of which problem 3 is an example), using the stochastic calculus. Problem 4 is the Dirichlet problem. Although this is purely deterministic we outline in Chapters VII and VIII how the introduction of an associated Ito diffusion (i.e. solution of a stochastic differential equation) leads to a simple, intuitive and useful stochastic solution, which is the cornerstone of stochastic potential theory. Problem 5 is (a discrete version of) an optimal stopping problem. In Chapter IX we represent the state of a game at time t by an Ito diffusion and solve the corresponding optimal stopping problem. The solution involves potential theoretic notions, such as the generalized harmonic extension provided by the solution of the Dirichlet problem in Chapter VIII. Problem 6 is a stochastic version of F. P. Ramsey's classical control problem from 1928. In Chapter X we formulate the general stochastic control problem in terms of stochastic differential equations, and we apply the results of Chapters VII and VIII to show that the problem can be reduced to solving the (deterministic) Hamilton-Jacobi-Bellman equation. As an illustration we solve a problem about optimal portfolio selection.

After the course was first given in Edinburgh in 1982, revised and expanded versions were presented at Agder College, Kristiansand and University of Oslo. Every time about half of the audience have come from the applied section, the others being so-called "pure" mathematicians. This fruitful combination has created a broad variety of valuable comments, for which I am very grateful. I particularly wish to express my gratitude to K. K. Aase, L. Csink and A. M. Davie for many useful discussions.

I wish to thank the Science and Engineering Research Council, U.K. and Norges Almenvitenskapelige Forskningsråd (NAVF), Norway for their financial support. And I am greatly indebted to Ingrid Skram, Agder College and Inger Prestbakken, University of Oslo for their excellent typing – and their patience with the innumerable changes in the manuscript during these two years.

Oslo, June 1985 Bernt Øksendal

Note: Chapters VIII, IX, X of the First Edition have become Chapters IX, X, XI of the Second Edition

We have not succeeded in answering all our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things.

Posted outside the mathematics reading room,
Tromsø University

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