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To

Professor Kazimierz Kuratowski

Preface

There are two aspects to the theory of Boolean algebras; the algebraic and the set-theoretical. A Boolean algebra can be considered as a special kind of algebraic ring, or as a generalization of the set-theoretical notion of a field of sets. Fundamental theorems in both of these directions are due to M. H. STONE, whose papers have opened a new era in the development of this theory.

This work treats the set-theoretical aspect, with little mention being made of the algebraic one.

The book is composed of two chapters and an appendix.

Chapter I is devoted to the study of Boolean algebras from the point of view of finite Boolean operations only; a greater part of its contents can be found in the books of BIRKHOFF [2] and HERMES [1]. Chapter II seems to be the first systematic study of Boolean algebras with infinite Boolean operations.

To understand Chapters I and II it suffices only to know fundamental notions from general set theory and set-theoretical topology. No knowledge of lattice theory or of abstract algebra is presumed. Less familiar topological theorems are recalled, and only a few examples use more advanced topological means; but these may be omitted. All theorems in both chapters are given with full proofs.

On the other hand, no complete proofs are given in the Appendix, which contains mainly a short exposition of some of the applications of Boolean algebras to other parts of mathematics with references to the literature. An elementary knowledge of the theories discussed is assumed.

I am very much indebted to Professor PAUL R. HALMOS for suggesting that I write this book.

I wish to express my thanks to H. BASS, A. BIAŁYNICKI-BIRULA and R. WHERRITT for the revision of the manuscript, and to J. BROWKIN, R. ENGELKING and T. TRACZYK for help in proofreading.

Warsaw-New Orleans-Princeton
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ROMAN SIKORSKI

Preface to the second edition

Chapter I and the Appendix are almost unchanged. On the contrary, many new results are included in Chapter II; some sections have been extended while others have been completely rewritten. However the general character of Chapter II has been preserved.

I am very grateful to PH. DWINGER, H. GAIFMAN, A. W. HALES, J. D. HALPERN, C. R. KARP, K. MATTHES, R. S. PIERCE, Z. SEMADENI and F. M. YAQUB for valuable information which helped greatly in bringing the material up to date.

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Aarhus, 1962

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