DIE GRUNDLEHREN DER

MATHEMATISCHEN WISSENSCHAFTEN

IN EINZELDARSTELLUNGEN MIT BESONDERER BERÜCKSICHTIGUNG DER ANWENDUNGSGEBIETE

BAND 121

SPRINGER-VERLAG
BERLIN · GÖTTINGEN · HEIDELBERG
1965

MARKOV PROCESSES

BY

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UNIVERSITY OF CALIFORNIA, BERKELEY

VOLUME I

1965

NEW YORK

ACADEMIC PRESS INC., PUBLISHERS

BERLIN · GÖTTINGEN · HEIDELBERG

SPRINGER-VERLAG
BERLIN · GÖTTINGEN · HEIDELBERG

Published in U. S. A. and Canada by
ACADEMIC PRESS INC., PUBLISHERS
111 Fifth Avenue, New York, N. Y. 10003

ISBN 978-3-662-00033-5 *ISBN 978-3-662-00031-1 (eBook)*
DOI 10.1007/978-3-662-00031-1

Library of Congress Catalog Card Number 64-24812

SOFTCOVER REPRINT OF THE HARDCOVER 1ST EDITION 1965

Preface

The modern theory of Markov processes has its origins in the studies of A. A. MARKOV (1906–1907) on sequences of experiments ,,connected in a chain" and in the attempts to describe mathematically the physical phenomenon known as Brownian motion (L. BACHELIER 1900, A. EINSTEIN 1905). The first correct mathematical construction of a Markov process with continuous trajectories was given by N. WIENER in 1923. (This process is often called the Wiener process.) The general theory of Markov processes was developed in the 1930's and 1940's by A. N. KOLMOGOROV, W. FELLER, W. DOEBLIN, P. LÉVY, J. L. DOOB, and others.

During the past ten years the theory of Markov processes has entered a new period of intensive development. The methods of the theory of semigroups of linear operators made possible further progress in the classification of Markov processes by their infinitesimal characteristics. The broad classes of Markov processes with continuous trajectories became the main object of study. The connections between Markov processes and classical analysis were further developed. It has become possible not only to apply the results and methods of analysis to the problems of probability theory, but also to investigate analytic problems using probabilistic methods. Remarkable new connections between Markov processes and potential theory were revealed. The foundations of the theory were reviewed critically: the new concept of strong Markov process acquired for the whole theory of Markov processes great importance.

This book attempts a systematic exposition of the modern theory of Markov processes. The newest directions, which have been barely treated in monographs, are given a great deal of attention.

A rigorous construction of the theory of Markov processes (as of the theory of stochastic processes in general) is impossible without recourse to a rather heavy set-theoretical apparatus. DYNKIN's monograph ,,Osnovaniia teorii markovskikh protsessov" (1959)* is devoted to the development of such apparatus and is a logical foundation for the present book. However, if the reader is willing to accept certain assertions on faith, he need not refer to the monograph, since all the necessary basic information is presented again. Nevertheless, whenever an assertion is

* Translators' note: The authorized German translation of this book was published in 1961 by Springer-Verlag under the title ,,Die Grundlagen der Theorie der Markoffschen Prozesse". Henceforth references will be made to this translation, in which all chapters, sections, theorems, etc. are numbered as in the original.

stated without proof, it is accompanied by an exact reference to the proof in „Die Grundlagen der Theorie der Markoffschen Prozesse".

The general theory of Markov processes, including the special case of processes homogeneous in time, was constructed in „Die Grundlagen der Theorie der Markoffschen Prozesse". Only homogeneous processes are considered in the present book. The fact is that most of the concrete results refer to the homogeneous case, and the extension to arbitrary processes, wherever possible, is easily accomplished by using the material developed in „Grundlagen".

To simplify the presentation we shall always assume that the time t varies in the interval $[0, \infty)$. The case where t assumes only integral values is much simpler, and all results which carry over to this case can be obtained without any difficulty.

The Introduction, which follows the Preface, summarizes the present state of the theory of Markov processes, its connections with mathematical analysis, and some unsolved problems*.

The contents of the book are divided into five principal parts. In the first, consisting of chapters 1—5, the general theory of homogeneous Markov processes is presented. Attention here is centered on infinitesimal and characteristic operators. The second part (chapters 6—11) is devoted to additive functionals and transformations of processes; in particular Itô's theory of stochastic integrals and stochastic integral equations is presented here. In the third part (chapters 12—13) harmonic and superharmonic functions related to a process are studied and probabilistic formulae for the solution of certain differential equations are derived. In the fourth part (chapter 14) the general results of the previous chapters are applied to the investigation of the n-dimensional Wiener process and its transformations. In the fifth and final part (chapters 15—17) continuous strong Markov processes on the line are studied.

The body of the book contains frequent references to the Appendix, which contains various information necessary for the understanding of the material in the main text. This includes certain results of measure theory, of probability theory and of the theory of differential equations. The Appendix contains all the necessary definitions, statements of theorems and references to books where proofs of these theorems can be found. The reader may find it convenient to glance through the Appendix and to refer to it whenever necessary in the course of reading the main text.

In the study of any mathematical theory, the proofs of various propositions demand differing degress of attention. There are proofs which

* In writing the Introduction, the author made use of the material of a survey lecture which he prepared for the International Congress of Mathematicians in Stockholm (August, 1962).

describe the essence of the situation better than the final formulations of results do. In order to master the subject, such proofs must be analyzed in detail. On the other hand, there are proofs that one hardly finds necessary to recollect in spite of the fact that the relevant results are frequently applied. Proofs of this second kind are marked in the book by the sign°. Some familiarity with them is useful for the acquisition of technique; however it is questionable whether it is worthwhile to spend much time on them.

An Historical-bibliographical Note, indicating the sources of the results presented in each chapter, is included at the end of the book; for each chapter, the Note contains a short survey of the literature on related questions. Some additional remarks at the end of the Note treat those areas of the theory of Markov processes which were not touched upon in the book.

Almost all references to the literature are collected in the Appendix and in the Historical-bibliographical Note. For each work, the author's name, the number in the Bibliography at the end of the book, and, as a rule, the year of publication are given. The monograph „Die Grundlagen der Theorie der Markoffschen Prozesse" frequently will be cited by the abbreviation GTMP.

Recently, several books devoted, wholly or partly, to new directions in the theory of Markov processes have appeared. KAI-LAI CHUNG's monograph (1960) on Markov processes with a countable state space gives a good presentation of the current state of this field. K. ITÔ's "Stochastic Processes" (1957) and M. LOÈVE's "Probability Theory" (second edition, 1960) are textbooks which contain large sections presenting the semigroup approach to the theory of Markov processes and investigating continuous processes on the line. (Our chapters 15—17 contain a more extensive theory of such processes.) A monograph by K. ITÔ and H. MCKEAN devoted to homogeneous continuous strong Markov processes is being prepared for publication. The author had the opportunity to see several chapters of this interesting book. It contains much concrete material about one-dimensional continuous processes and a short survey of the theory of n-dimensional processes, of which only the Wiener process is examined in detail.

Throughout a number of years at Moscow University, a seminar on the theory of Markov processes has been working under the direction of the author. The results obtained by its participants occupy a significant part of this book. The discussions at the meetings of the seminar were extremely useful to the author. I take this opportunity to express my gratitude to the members of the seminar: A. D. WENTZELL, V. A. VOLKONSKI, I. V. GIRSANOV, L. V. SEREGIN, V. N. TUTUBALIN, M. I. FREIDLIN, R. Z. KHASMINSKII, M. G. SHUR, A. A. IUSHKEVICH, and others.

I am especially indebted to A. A. IUSHKEVICH, who carefully read through the entire manuscript and noted a number of places needing clarification and revision.

I consider it my pleasant duty to thank O. A. OLEINIK and A. S. KALASHINKOV for valuable consultations on the theory of partial differential equations.

Also, I want to thank I. L. GENIS and O. S. KONSTANTINOVA for their work on the technical preparation of the manuscript.

March, 31, 1962 E. B. DYNKIN

Preface to the English edition

This book is devoted to the modern theory of Markov processes. Recent research has led to the creation of various analytical tools and techniques (differential and stochastic integral equations, infinitesimal and characteristic operators, additive functionals, potentials and superharmonic functions connected with Markov processes) and to a better insight into the structure of large classes of Markov processes (one-dimensional continuous processes, diffusion processes, generalized Brownian motion, etc.). The presentation of these new results is the main purpose of the book.

The necessary set-theoretic background was studied by the author in his book "Foundations of the theory of Markov processes" (Moscow, 1959), of which the German translation appeared in 1961 in this same series (Die Grundlagen der mathematischen Wissenschaften, Band 108). No previous knowledge of this book is required, since the necessary results are summarized here. However, the reader who wishes to become familiar with the proofs of the quoted theorems will have to turn to the corresponding sections of "Foundations of the theory of Markov processes".

The English translation was prepared by a group of young mathematicians working at the Statistical Laboratory of the University of California at Berkeley: Mr. JAAP FABIUS, Miss VIDA GREENBERG, Mr. ASHOK MAITRA and Mr. GIANDOMENICO MAJONE. Professor JERZY NEYMAN acted as organizer and adviser to this group and maintained close contact with the translators and the author. Not a single doubtful point has been settled without his active participation.

The entire text of the translation has been carefully read by the author.

Expressing deep gratitude to Professor NEYMAN and to the translators, the author hopes that the English translation published by Springer-Verlag may further the development of international scientific relations in the field of mathematics.

Moscow, March 10, 1963 E. B. DYNKIN

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