



Sylvestre Gallot
Dominique Hulin
Jacques Lafontaine

Riemannian Geometry

Second Edition
With 95 Figures

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Sylvestre Gallot
Ecole Polytechnique, Centre de Mathématiques
Unité de Recherche Associée du CNRS D 0169
F-91128 Palaiseau Cedex, France

Dominique Hulin
Ecole Polytechnique, Centre de Mathématiques
Unité de Recherche Associée du CNRS D 0169
F-91128 Palaiseau Cedex, France

Jacques Lafontaine
Université de Montpellier
Département de Mathématiques
GETODIM – Unité de Recherche Associée du CNRS 1407
F-34095 Montpellier Cedex 5, France

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PREFACE TO THE SECOND EDITION

In this second edition, the main additions are a section devoted to surfaces with constant negative curvature, and an introduction to conformal geometry.

Also, we present a -soft- proof of the Paul Levy-Gromov isoperimetric inequality, kindly communicated by G. Besson.

Several people helped us to find bugs in the first edition. They are not responsible for the persisting ones! Among them, we particularly thank Pierre Arnoux and Stefano Marchiafava.

We are also indebted to Marc Troyanov for valuable comments and suggestions.

INTRODUCTION

This book is an outgrowth of graduate lectures given by two of us in Paris. We assume that the reader has already heard a little about differential manifolds. At some very precise points, we also use the basic vocabulary of representation theory, or some elementary notions about homotopy. Now and then, some remarks and comments use more elaborate theories. Such passages are inserted between *.

In most textbooks about Riemannian geometry, the starting point is the local theory of embedded surfaces. Here we begin directly with the so-called “abstract” manifolds. To illustrate our point of view, a series of examples is developed each time a new definition or theorem occurs. Thus, the reader will meet a detailed recurrent study of spheres, tori, real and complex projective spaces, and compact Lie groups equipped with bi-invariant metrics. Notice that all these examples, although very common, are not so easy to realize (except the first) as Riemannian submanifolds of Euclidean spaces.

The first chapter is a quick introduction to differential manifolds. Proofs are often supplied with precise references. However, numerous examples and exercises will help the reader to get familiar with the subject.

Chapters II and III deal with basic Riemannian geometry of manifolds, as described in the content table. We finish with global results (Cartan-Hadamard, Myers’ and Milnor’s theorems) concerning relations between curvature and topology. By the way, we did not resist the temptation to give an overview of recent research results. We hope the reader will want to look at the original papers.

Analysis on manifolds has become a wide topic, and chapter IV is only an introduction. We focused on the Weitzenböck formula, and on some aspects of spectral theory. Our Ariadne’s thread was what our “Mentor” Marcel Berger calls “la domination universelle de la courbure de Ricci”, discovered by Gromov in the seventies. Chapter V deals with more classical topics in Riemannian submanifolds.

The reader will find numerous exercises. They should be considered as a part of the text. This is why we gave the solutions of most of them.

SOME HISTORICAL AND HEURISTIC REFERENCES

Riemannian Geometry is indeed very lively today, but most basic notions go back to the nineteenth century.

At the beginning of each chapter, we endeavored to explain what is going on. But these short introductions do not replace serious history or popularization works. We recommend the following, to be read concurrently with our book:

- E. Cartan : La Géométrie des espaces de Riemann [Ca];
- M. Spivak : Differential geometry (t.2) [Sp];
- P. Dombrowski : 150 years after Gauss' "disquisitiones generales circa superficies curvas" [Dom];
- M. Berger : La géométrie métrique des variétés riemanniennes [Br3];
- J.P. Bourguignon : Géométrie et Physique [Bn];
- S. Gallot : Géométries [Ga5].

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CONTENTS

Chapter I. Differential Manifolds

A. From Submanifolds to Abstract Manifolds

Submanifolds of \mathbf{R}^{n+k}	2
Abstract manifolds	6
Smooth maps	11

B. Tangent Bundle

Tangent space to a submanifold of \mathbf{R}^{n+k}	13
The manifold of tangent vectors	14
Vector bundles	16
Differential map	17

C. Vector Fields

Definitions	18
Another definition for the tangent space	19
Integral curves and flow of a vector field	23
Image of a vector field under a diffeomorphism	24

D. Baby Lie Groups

Definitions	27
Adjoint representation	29

E. Covering Maps and Fibrations

Covering maps and quotient by a discrete group	29
Submersions and fibrations	31
Homogeneous spaces	33

F. Tensors

Tensor product (digest)	36
Tensor bundles	36
Operations on tensors	37
Lie derivatives	39
Local operators, differential operators	40
A characterization for tensors	40

G. Exterior Forms

Definitions	42
Exterior derivative	43
Volume forms	46
Integration on an oriented manifold	47
Haar measure on a Lie group	48
H. Appendix: Partitions of Unity	48

Chapter II. Riemannian Metrics**A. Existence Theorems and First Examples**

Definitions	52
First examples	54
Examples: Riemannian submanifolds, product Riemannian manifolds	58
Riemannian covering maps, flat tori	59
Riemannian submersions, complex projective space	63
Homogeneous Riemannian spaces	65

B. Covariant Derivative

Connections	69
Canonical connection of a Riemannian submanifold	72
Extension of the covariant derivative to tensors	73
Covariant derivative along a curve	75
Parallel transport	77
Examples	78

C. Geodesics

Definitions	80
Local existence and uniqueness for geodesics, exponential map	83
Riemannian manifolds as metric spaces	87
Complete Riemannian manifolds, Hopf-Rinow theorem	94
Geodesics and submersions, geodesics of $P^n\mathbf{C}$	97
Cut locus	100

Chapter III. Curvature**A. The Curvature Tensor**

Second covariant derivative	107
Algebraic properties of the curvature tensor	108
Computation of curvature: some examples	109
Ricci curvature, scalar curvature	111

B. First and Second Variation of Arc-Length and Energy

Technical preliminaries:	
vector fields along parameterized submanifolds	112

First variation formula	114
Second variation formula	116
C. Jacobi Vector Fields	
Basic topics about second derivatives	118
Index form	119
Jacobi fields and exponential map	121
Applications: S^n , H^n , $P^n\mathbf{R}$, 2-dimensional Riemannian manifolds	122
D. Riemannian Submersions and Curvature	
Riemannian submersions and connections	124
Jacobi fields of $P^n\mathbf{C}$	125
O'Neill's formula	127
Curvature and length of small circles. Application to Riemannian submersions	128
E. The Behavior of Length and Energy in the Neighborhood of a Geodesic	
The Gauss lemma	130
Conjugate points	131
Some properties of the cut-locus	134
F. Manifolds with Constant Sectional Curvature	
Spheres, Euclidean and hyperbolic spaces	135
G. Topology and Curvature	
The Myers and Hadamard-Cartan theorems	137
H. Curvature and Volume	
Densities on a differentiable manifold	139
Canonical measure of a Riemannian manifold	140
Examples: spheres, hyperbolic spaces, complex projective spaces	142
Small balls and scalar curvature	143
Volume estimates	144
I. Curvature and Growth of the Fundamental Group	
Growth of finite type groups	148
Growth of the fundamental group of compact manifolds with negative curvature	149
J. Curvature and Topology: An Account of Some Old and Recent Results	
Introduction	151
Traditional point of view: pinched manifolds	151
Almost flat pinching	153
Coarse point of view: compactness theorems of Cheeger and Gromov	153

K. Curvature Tensors and Representations of the Orthogonal Group

Decomposition of the space of curvature tensors	154
Conformally flat manifolds	157
The second Bianchi identity	158

L. Hyperbolic Geometry

Introduction	159
Angles and distances in the hyperbolic plane	159
Polygons with “many” right angles	164
Compact surfaces	166
Hyperbolic trigonometry	168
Prescribing constant negative curvature	172

M. Conformal Geometry

Introduction	174
The Moebius group	174
Conformal, elliptic and hyperbolic geometry	177

Chapter IV. Analysis on Manifolds and the Ricci Curvature

A. Manifolds with Boundary

Definition	181
The Stokes theorem and integration by parts	182

B. Bishop’s Inequality Revisited

Some commutations formulas	185
Laplacian of the distance function	186
Another proof of Bishop’s inequality	187
The Heintze-Karcher inequality	188

C. Differential Forms and Cohomology

The de Rham complex	190
Differential operators and their formal adjoints	190
The Hodge-de Rham theorem	193
A second visit to the Bochner method	194

D. Basic Spectral Geometry

The Laplace operator and the wave equation	196
Statement of the basic results on the spectrum	198

E. Some Examples of Spectra

Introduction	199
The spectrum of flat tori	200
Spectrum of (S^n, can)	201

F. The Minimax Principle

The basic statements 203

G. The Ricci Curvature and Eigenvalues Estimates

Introduction 207
 Bishop's inequality and coarse estimates 207
 Some consequences of Bishop's theorem 208
 Lower bounds for the first eigenvalue 210

H. Paul Levy's Isoperimetric Inequality

Introduction 212
 The statement 212
 The proof 213

Chapter V. Riemannian Submanifolds

A. Curvature of Submanifolds

Introduction 217
 Second fundamental form 217
 Curvature of hypersurfaces 219
 Application to explicit computations of curvatures 221

B. Curvature and Convexity

The Hadamard theorem 224

C. Minimal Surfaces

First results 227

Some Extra Problems 232

Solutions of Exercises

Chapter I 234
 Chapter II 244
 Chapter III 261
 Chapter IV 266
 Chapter V 268

Bibliography 272

Index 279