

Rolf Nevanlinna

Analytic Functions

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Phillip Emig



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Preface

The present monograph on analytic functions coincides to a large extent with the presentation of the modern theory of single-valued analytic functions given in my earlier works "Le théorème de Picard-Borel et la théorie des fonctions méromorphes" (Paris: Gauthier-Villars 1929) and "Eindeutige analytische Funktionen" (Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen, Vol. 46, 1st edition Berlin: Springer 1936, 2nd edition Berlin-Göttingen-Heidelberg Springer 1953).

In these presentations I have strived to make the individual results and their proofs readily understandable and to treat them in the light of certain guiding principles in a unified way. A decisive step in this direction within the theory of entire and meromorphic functions consisted in replacing the classical representation of these functions through canonical products with more general tools from the potential theory (Green's formula and especially the Poisson-Jensen formula). On this foundation it was possible to introduce the quantities (the characteristic, the proximity and the counting functions) which are definitive for the description of the asymptotic properties of an analytic function in the vicinity of essential singularities. At the same time they lead to far-reaching extensions and sharpenings of Picard's theorem, in the direction of the general value distribution theory, which is concerned with the distribution and density of those points at which an analytic function assumes a preassigned value, whereby all complex values are to be considered. This change in method has also led to new insights in another direction: it has helped to bring the algebraic-analytic points of view basic to the Cauchy-Weierstrass function theory into closer contact with the principles of the Riemannian function theory, which in greater measure emphasizes the potential theoretical and geometrical features of the analytic mappings under study.

This new edition contains some changes and additions, particularly concerning the Second Main Theorem. At the suggestion of several colleagues, I have included my "elementary" proof of the theorem. I also give a version of F. NEVANLINNA's differential geometrical method which makes the main theorem easier of access.

In the course of the last three decades the literature on the topics treated here and related questions has grown enormously. Of the works

which contain important contributions to these problems I wish to mention in particular:

WEYL, H.: *Meromorphic Curves and Analytic Curves*. Princeton: Princeton University Press 1943.

HAYMAN, W. K.: *Meromorphic Functions*. Oxford: Clarendon Press 1964.

WITTICH, H.: *Neuere Untersuchungen über eindeutige analytische Funktionen*. Berlin-Göttingen-Heidelberg: Springer 1955.

This book has been translated into English with great care and interest by Dr. P. EMIG. For this I express to him my sincere thanks. I am also indebted to Cand. Phil. T. HEIKKURINEN for her help in the reading and correction of the proofs. Finally, I should like to express my thanks to Springer-Verlag for their friendly cooperation in the production of this volume.

Helsinki, February 1970

ROLF NEVANLINNA

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