The Life Table

Mathematical demography has its modern beginnings in the gradual development of correct procedures for forming life tables, and in a single remarkable paper by Leonard Euler (1760, paper 11 below) that introduced stable age distributions. One far older work of at least some quality survives: a third century a.d. table of life expectancies attributed to Ulpian (paper 1) that remained in use in northern Italy through the 18th century. The table and an accompanying discussion have been taken from C.F. Trenerry (1926).

The mathematician Girolamo Cardano took up the problem briefly in 1570, but without substantive results. Cardano made the assumption that a man who took great care in all things would have a certain life expectancy α, so that $\mathring{e}_x = \alpha - x$ for all ages x, and then asked what part of this might be forfeited by a relaxation of prudence. He proposed letting life expectancy fall by $\frac{1}{40}$ of its value during each year in which a man was reasonably careful but not fastidious: by the nature of the life expectancies, a man might be born with the prospect of living say 260 years and yet die at age 80, having every year thrown away by inattention a part of what remained to him (Cardano 1570, pp. 204—211). A modern interpretation would be that a cohort carries its life table with it. The result was not generalized to populations.

John Graunt's *Natural and Political Observations Upon the Bills of Mortality* (1662) is the first substantive demographic work to have been written. The book is occasionally curious but most often impressive, even from a perspective of three hundred years: Graunt culled a remarkable amount of information from the christening and death lists begun in the later plague period and usually understood its implications. Parts of the treatise are included here as paper 2.

A second work of great importance followed upon Graunt's, Edmund Halley's (1693) presentation of the Breslau (Wroclaw Poland) life table (paper 3). Halley had made an effort to obtain the Breslau lists in order to see what might be done with them, after learning of their apparent quality.

The methods of calculation Halley used in his life table were partly informal, as in his remarks on stationarity and in his unorthodox subtraction (where l_x is the number of survivors to exact age x in the life table from among l_0 births and L_x represents the number between ages x and $x+1$)

$$890\,[=l_1] - 198 = 692\,[=L_6]\,,$$

explained by the oblique statement: "198 do die in the *Five Years* between 1 and 6 compleat, taken at a *Medium*." [The terminology has created confusion down to the present century. Raymond Pearl (1922, p. 83), apparently reading the L_x terms that make up Halley's table as l_x, calculated life expectancy at birth as 33.5 years by the table instead of the correct 27.5. The mistake is carried over in Dublin, Lotka and Spiegelman (1949, p. 34)].

Johan DeWit (1671) preceded Halley in the correct calculation of annuities, using exact (l_x) as against Halley's approximate (L_x) denominators, and most of Halley's other *Uses* can be answered differently, but the quality of Halley's table and discussion much surpasses the few earlier works and several of the subsequent ones. His table is graphed below, alongside Ulpian's, Graunt's, DeWit's, and as references the middle level table for Crulai c. 1700 (Gautier and Henry 1958, pp. 163, 190) and one of the Coale and Demeny (1966) model life tables.

After Halley, the next impressive contribution was the series of life tables for annuitants and monastic orders by Antoine Deparcieux, printed in 1746. The accuracy of Deparcieux's data was sufficient for him to show that adult life expectancies had been increasing over the previous half century. Deparcieux calculated his $\mathring{e}_x$ values by the simple but adequate formula

$$\mathring{e}_x = \frac{\sum_{i=x}^{\omega} (l_i - l_{i+1})(i + 0.5 - x)}{l_x} = \frac{\sum_{i=x}^{\omega} l_i}{l_x} - 0.5 \, .$$

Of their utility he writes (1760, pp. 58—59): "Les vies moyennes [i.e., $\mathring{e}_x$] sont ce qui m'a paru de plus commode pour faire promptement & sans aucun calcul, la comparaison des différent ordres de mortalité qu'on a établis ... [Life expectancies are what have appeared to me most convenient for making promptly and without any calculation a comparison of different orders of mortality that one has established]."

Two later efforts merit attention here: Daniel Bernoulli (1766) introduced continuous analysis and suggested the force of mortality [$\mu(x)$] in an application of differential calculus to the analysis of smallpox rates. Later Émmanuel Étienne Duvillard (1806), in an article that also introduced the T_x column (defined as $T_x = \sum_x^{\omega} L_i$), applied Bernoulli's method to estimate the increase in life expectancy that would follow if smallpox were eliminated by Edward Jenner's vaccine. The calculus, which these and most modern work employ, dates to a seventy year period (1665—1736) between Isaac Newton's first investigations and the publication of his principal works. Westergaard (1969, pp. 92—93) comments however that it was not until the late nineteenth century that continuous analysis was widely enough understood for Bernoulli's work to be appreciated.

Joshua Milne in his excellent *Treatise on the Valuation of Annuities* (1815), which includes a careful analysis of life tables made prior to his, was first to suggest a formula by which l_x values could be calculated for real populations.

His is the well known expression

$$d_x = l_x\left(\frac{D_x}{P_x + \frac{1}{2}D_x}\right)$$

where d_x is the number dying between ages x and $x+1$ in the life table population, D_x represents calendar year deaths between these ages in an observed population, and $P_x + \frac{1}{2}D_x$ constructs an initial population analogous to l_x except for scale by adding half of the yearly deaths to the observed midyear population P_x.

Excerpts from Milne's discussions of the life table and of age-specific fertility rates (due to Henric Nicander (1800, pp. 323—324)) are given in paper 4. Milne's method for graduating data from grouped to single ages, not the best, has been omitted. His footnoted criticism of Thomas Simpson's (1742) work opens an area of discussion that is easily missed: Milne's life table was misread by William Sutton (1884)—whose clarification in 1874 of the construction of Richard Price's 1771 Northampton Table is a more competent work—but was immediately re-established by George King (1884). Like other fields, demography does not only move forward.

The fifth article in this section excerpts from George King (1902), whose notation is contemporary, his explanations of terms of the life table. From the middle of the 19th century William Farr standardized much of the life table, but he did not put his work in a form at all comparable to King's excellent text-book. [A recent addition to the life table, from C. L. Chiang (1960a), is the term ${}_na_x$. This is King's unremembered "average amount of existence between ages x and $x+n$, belonging to those who die between these ages," i. e.: ${}_na_x = \frac{{}_nL_x - n\,l_{x+n}}{{}_nd_x}$.]

Out of sequence, the Lexis (1875) diagram is introduced in paper 6. For most of a century it has been a standby of all analysis attempting to relate age and time. Among contemporary works, those of Roland Pressat (1969, 1972) exploit it most fully.

The important contributions to the life table in this century have been competent abridgement techniques for generating tables by five or ten year age groupings in place of single years of age. The Lowell Reed and Margaret Merrell (1939) article included here as paper 7 did much to establish the validity of abridgement techniques by its introduction of an attractive expression:

$${}_nq_x = 1 - \exp[-n\,{}_nm_x - a\,n^3\,{}_nm_x^2]$$

for estimating ${}_nq_x$ from ${}_nm_x$ values where wide age groupings are used. In the expression, ${}_nm_x$ is the age-specific death rate in the life table population ages x to $x+n$ (that is, ${}_nm_x = {}_nd_x/{}_nL_x$) and ${}_nq_x$ the probability of dying within the interval for a person of exact age $x({}_nq_x = {}_nd_x/l_x)$. By empirical examination the authors found that the constant a required by the expression could be the same for all ages above infancy. Reed and Merrell examine two other approximations to

${}_nq_x$, the first of which:

$$ {}_nq_x = \frac{n\,{}_nm_x}{1 + \frac{n}{2}\,{}_nm_x} $$

can be derived from Milne's formula for d_x; the other due apparently to Farr (1864, pp. xxiii—xxiv), and evident earlier to Gompertz (1825, paper 30 below):

$$ {}_nq_x = 1 - \exp[-n\,{}_nm_x] \,. $$

Following their article we include derivations for both expressions.

T.N.E. Greville (1943) was able through ingenious expansions to derive each of these equations by working with the definitions of ${}_nm_x$ and ${}_nq_x$ and to show that the Reed-Merrell formula incorporates Gompertz' Law that the force of mortality is an exponential function of age. The assumption is appropriate at older ages and inappropriate for infancy, and thus defines the age range over which the Reed-Merrell formula is applicable. In the same article Greville discusses approximations to ${}_nL_x$ values where, as before, age groupings are wide (paper 8).

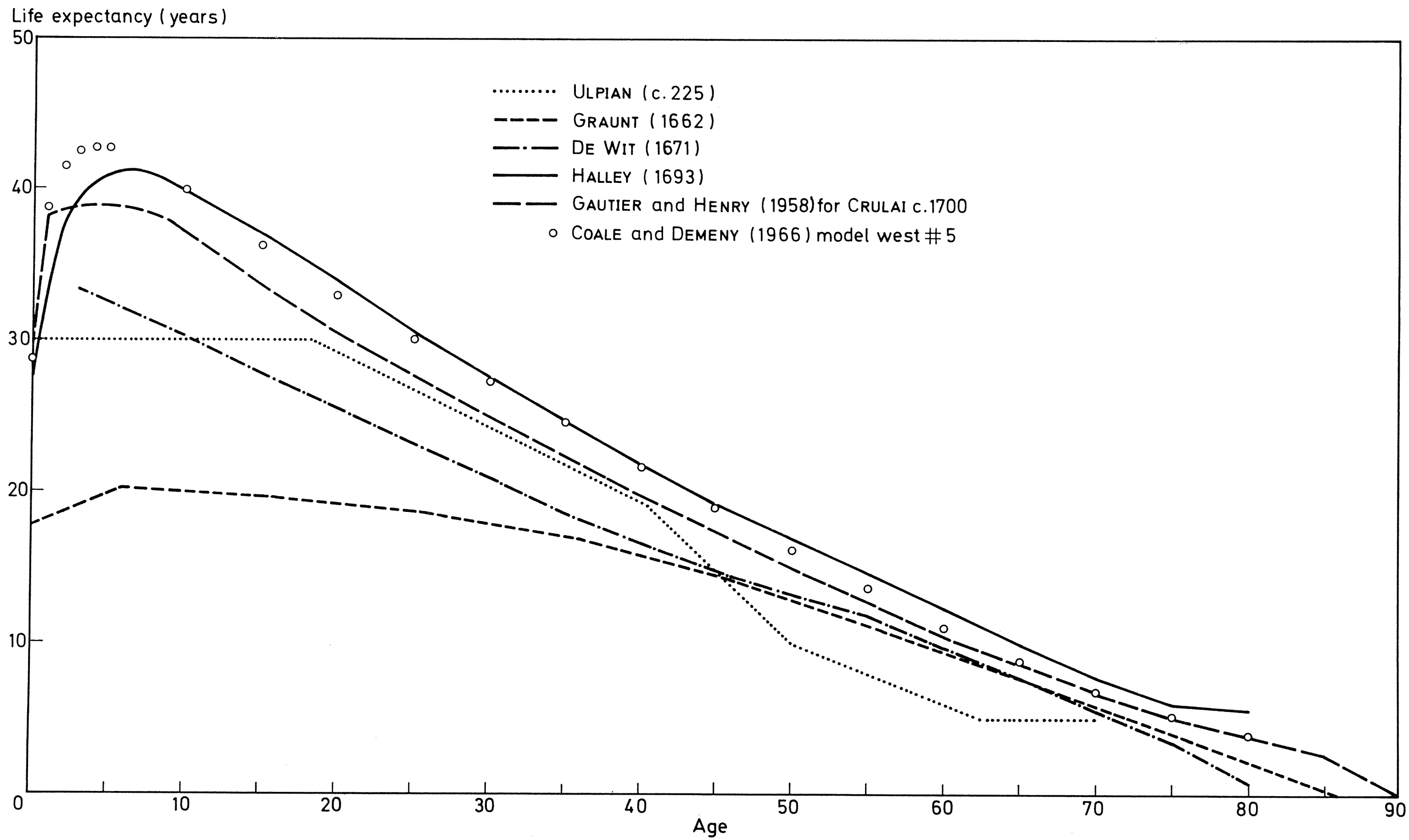
The methods used by Greville can be generalized to take advantage of the observed age structure of a population as well as its mortality schedule. In the simplest case this gives rise to the formula, due to Nathan Keyfitz and James Frauenthal (1975),

$$ {}_nq_x = 1 - \exp\left[-n\,{}_nM_x + \frac{n}{48\,{}_nP_x}({}_nP_{x+n} - {}_nP_{x-n})({}_nM_{x+n} - {}_nM_{x-n}) \right] $$

with as before the caveat that infancy requires separate consideration.

The chapter concludes with excerpts from the well known article by Edward Deevey (1947) in which he evaluates efforts that had been made up to that time to develop life tables for animal populations.

Sampling variances of life table terms are taken up in chapter 7.


Life expectancy (years)
50
40
30
20
10
0
10
20
30
40
50
60
70
80
90
Age
Ulpian (c. 225)
Graunt (1662)
De Wit (1671)
Halley (1693)
Gautier and Henry (1958) for Crulai c.1700
Coale and Demeny (1966) model west # 5