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Potential Theory

An Analytic and Probabilistic Approach to Balayage

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To Brigitte and Luise

Contents

Introduction

During the last thirty years potential theory has undergone a rapid development, much of which can still only be found in the original papers. This book deals with one part of this development, and has two aims. The first is to give a comprehensive account of the close connection between analytic and probabilistic potential theory with the notion of a balayage space appearing as a natural link. The second aim is to demonstrate the fundamental importance of this concept by using it to give a straight presentation of balayage theory which in turn is then applied to the Dirichlet problem. We have considered it to be beyond the scope of this book to treat further topics such as duality, ideal boundary and integral representation, energy and Dirichlet forms.

The subject matter of this book originates in the relation between classical potential theory and the theory of Brownian motion. Both theories are linked with the Laplace operator. However, the deep connection between these two theories was first revealed in the papers of S. KAKUTANI [1], [2], [3], M. KAC [1] and J.L. DOOB [2] during the period 1944-54: This can be expressed by the·fact that the harmonic measures which occur in the solution of the Dirichlet problem are hitting distributions for Brownian motion or, equivalently, that the positive hyperharmonic functions for the Laplace equation are the excessive functions of the Brownian semigroup. This equivalence allows potential theoretic results and notions (such as balayage, fine topology, polar set, thinness and regular point) to be given a probabilistic interpretation.

This equivalence also led J.L. DOOB [3],[4],[5] to treat the Dirichlet problem for the heat equation using a combination of analytic and probabilistic methods. Based on these results and earlier attempts by G. TAUTZ [1] , [2] , M. BRELOT [9]-[12] and H. BAUER [3],[5] developed the potential theory of harmonic spaces. Starting from basic properties of harmonic functions (sheaf property, local solvability of the Dirichlet problem, convergence properties) an extensive theory was built up which covers a wide class of linear second order elliptic and parabolic partial differential equations. In particular, a common treatment of the Dirichlet problem for arbitrary open sets was given. This development culminated in the monograph of C. CONSTANTINESCU - A. CORNEA [4]. Meanwhile, on the probabilistic side, chapter V we obtain further examples by studying subspaces, subordination by convolution semigroups, products, and images. In particular, our list of standard examples is completed with Riesz potentials and the heat equation.

The second part of the book proper starts with chapter VI. Balayage of functions and balayage of measures are of central interest here. All proofs are given within the framework of balayage spaces, i.e. the proofs are analytic. However, all important notions are also interpreted probabilistically; we hope this will widen the appeal of the theory, and at the same time lead to a deeper understanding of many of the statements. Of course, it will be noted that even many analytic proofs are (and perhaps have to be) guided by probabilistic intuition. The interested reader is encouraged to find genuine probabilistic proofs of some of the results in order to see whether analytic subtleties can be avoided by considering suitable sets of paths and applying the strong Markov property. As an application of balayage theory different types of Dirichlet problems are studied in chapter VII. First the method of Perron-Wiener-Brelot is adapted to balayage spaces - where of course functions on the boundary of an open set have to be replaced by functions on its complement. The rich structure of the cone of continuous real potentials allows us to develop a Choquet type theory for cones of continuous superharmonic functions. This leads to a solution of the weak Dirichlet problem and yields important approximation theorems. The final chapter establishes that nice linear second order elliptic or parabolic differential operators generate harmonic spaces and that consequently their potential theory is covered by the present general theory.

Nearly all of the sections contain exercises. Their principal aim is to increase the familiarity of the reader with the material. We have added bibliographical notes including some historical remarks which are far from being complete. We apologize for all omissions and inaccuracies. For the convenience of the reader the last page of the book is a guide to our standard examples.

The material for this book has evolved out of courses for graduate students at the universities of Frankfurt and Bielefeld. The reader is assumed to be familiar with basic facts from functional analysis (e.g. Hahn-Banach theorem), measure theory (e.g. Radon measures on locally compact spaces), and probability theory (e.g. conditional expectations). However, no knowledge of potential theory is presupposed. Nevertheless, we expect that the book has something to offer to the expert, whether it be through simplified proofs or even because of the results themselves.

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fundamental papers of G.A. HUNT [1],[2],[3] during 1957-58 marked the beginning of a potential theory for Markov processes. A compact presentation of this theory can be found in the book of R.M. BLUMENTHAL - R.K. GETOOR [1].

Papers of P.A. MEYER [1] in 1963 and N. BOBOC - C. CONSTANTINESCU - A. CORNEA [1] in 1967 showed that, by analogy with the relation between classical potential theory and Brownian motion, every harmonic space admits corresponding Markov processes. But it was only in 1978 that J. BLIEDTNER - W. HANSEN [5] could characterize the class of Markov processes which are associated with harmonic spaces (see also J.C. TAYLOR [5]).

The introduction of harmonic spaces was motivated by the properties of solutions of certain differential equations. This is reflected in the fact that the corresponding Markov processes always have continuous paths, i.e. they are diffusions. However, whether the paths are continuous or not plays no important rôle in the potential theory of Markov processes. Moreover, even within the theory of harmonic spaces itself it turned out that potentials are more important than harmonic functions and that the really crucial properties of the cone of continuous potentials hold in a far more general setting.

This observation led the authors to combine investigations of G. MOKOBODZKI - D. SIBONY [1]-[4] with approaches used in the book of C. CONSTANTINESCU - A. CORNEA [4] and to introduce the notion of a balayage space. Not only does this concept elucidate the connections between analytic and probabilistic potential theory, it also allows a clear and direct presentation of balayage theory. This theory originates in H. POINCARÉ's method of balayage (balayer (french) = to sweep) to study the equilibrium problem now solved by M. BRELOT's reduction technique. All this can be carried out in the general framework of a balayage space without losing known results for harmonic spaces and without more complicated proofs. In particular, different types of Dirichlet problems can be treated in an elegant manner. A decisive additional advantage of this approach is that Riesz potentials and Markov chains become further standard examples and their potential theory can thus be covered.

In chapter 0 we give a concise presentation of the fact that the excessive functions for the Brownian semigroup are the same as the positive hyperharmonic functions for the Laplace operator. This discussion serves to motivate the general treatment of excessive functions for sub-Markov semigroups (chapter II) and of positive hyperharmonic functions for a family of harmonic kernels (chapter III). The necessary prerequisites from functional analysis can be found in chapter I. The first part of the book closes with a short introduction to Markov processes (chapter IV). The setup here is essentially geared to the main result, which shows that the various descriptions of potential theory using either balayage spaces, families of harmonic kernels, sub-Markov semigroups, or Markov processes are all equivalent. For better insight three standard examples are treated along the way, namely classical potential theory, translation in $\mathbf{R}$, and discrete potential theory. In

Basic Notations

$\mathbb{N}$	set of natural integers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{Z}$	set of integers
$\mathbb{Q}$	set of rational numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+, \mathbb{R}_+^*$	set of real numbers ≥ 0 , > 0
$\overline{\mathbb{R}}$	extended real line
$<\cdot,\cdot>, \|\cdot\|$	scalar product, norm on $\mathbb{R}^n$
λ^n	Lebesgue measure on $\mathbb{R}^n$
$\lim_{s\downarrow t}$ ($\lim_{s\uparrow t}$)	$\lim_{s\to t, s>t}$ ($\lim_{s\to t, s<t}$)
X	locally compact space with countable base
$\bar{A}, \mathring{A}, A^*$	closure, interior, boundary of A
$\mathfrak{P}(X)$	set of subsets of X
$\mathcal{U}$	base of relatively compact open subsets of X
$\mathcal{B}(X)$	set of all Borel numerical functions on X (subsets of X)
l.s.c. (u.s.c.)	lower (upper) semicontinuous
$\mathcal{C}(X)$	space of continuous real functions on X
$\mathcal{C}_0(X)$	space of continuous real functions on X vanishing at infinity
$\mathcal{K}(X)$	space of continuous real functions on X with compact support
$\mathcal{A}_b, \mathcal{A}^+, \mathcal{A}_r$	set of all bounded, positive, real functions in $\mathcal{A}$
$\|f\|$	uniform norm of $f \in \mathcal{B}_b(X)$
$\{f \geq g\}$	$\{x \in X : f(x) \geq g(x)\}$
$S(f)$	support of f, i.e. $\overline{\{f \neq 0\}}$
1_A	characteristic function of a set A
$f_{\|A}$	restriction of f on A or $1_A f$
$f \in \mathcal{A}/\mathcal{B}$	f is $\mathcal{A}$-$\mathcal{B}$-measurable
$\mathcal{M}_+(X)$	set of all (positive Radon) measures on X
$\mathcal{M}_+^1(X)$	set of all probability measures on X
supp (μ)	support of a measure μ
$\mu_{\|A}$	restriction of μ on A or $1_A\mu$
ε_x	unit mass (Dirac measure) at x