Part II. Geometric Theory of Elliptic Solutions of Monge–Ampere Equations

Introduction

Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be Cartesian coordinates in the $(n+1)$-dimensional Euclidean space E^{n+1}. We denote by E^n the hyperplane $x_{n+1} = 0$. Below we use the notation

$$x_{n+1} = z$$

and call

$$x = (x_1, x_2, \ldots, x_n) \quad \text{and} \quad (x, z) = (x_1, x_2, \ldots, x_n, z)$$

points of E^n and E^{n+1}.

Let G be a bounded domain in E^n and $C^2(G)$ be the set of C^2-functions defined in G. Let

$$H(u) = \det(u_{ij})^{*)} \tag{II.1}$$

for all $u(x) \in C^2(G)$. Then the operator

$$H\colon\ C^2(G) \to C(G)$$

is called the *simplest n-dimensional Monge–Ampere operator* or more briefly the *Monge–Ampere operator*. Partial differential equations containing $H(u)$ as the principal term are called the *Monge–Ampere equations*. More precisely, these equations can be described in the following way:

(a) Classical Monge–Ampere Equations ($n = 2$). Classical Monge–Ampere equations are related to functions with two independent variables x_1 and x_2. These equations have the form

$$u_{11}u_{22} - u_{12}^2 = Au_{11} + 2Bu_{12} + Cu_{22} + D, \tag{II.2}$$

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*) We use the notations: $u_i = u_{x_i}$, $u_{ij} = u_{x_i x_j}$, $i, j = 1, 2, \ldots, n$ and $\det(u_{ij}) =$

$$\det\begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ u_{n1} & u_{n2} & \cdots & u_{nn} \end{pmatrix}.$$

where A, B, C, D are given functions of x_1, x_2, u, u_1, u_2. The expression

$$\Delta = D + AC - B^2 \tag{II.3}$$

is called the discriminant of equation (II.2). For any solution $u(x_1, x_2) \in C^2(G)$ of equation (II.2) the identity

$$(u_{11} - C)(u_{22} - A) - (u_{12} + B)^2 = D + AC - B^2 \tag{II.4}$$

holds. This identity yields the ellipticity (hyperbolicity) of equation (II.2) if and only if $\Delta > 0$ $(\Delta < 0)$ for all $(x_1, x_2) \in G$, $u \in R$, $(u_1, u_2) \in R^2$.

Even the theory of the simplest classical Monge–Ampere equations

$$u_{11}u_{22} - u_{12}^2 = D, \qquad A = B = C = 0 \tag{II.5}$$

is rich and has various deep applications to global differential geometry, linear and quasilinear PDE, calculus of variations, applied mathematics and others.

All solutions of elliptic equations (II.5) are necessarily convex or concave functions. All solutions of hyperbolic equations (II.5) necessarily have saddle graphs.

(b) The n-Dimensional Simplest Monge–Ampere Equations. These equations have the form

$$\det(u_{ij}) = D(x, u, \text{grad } u). \tag{II.6}$$

Here the relationship between the graphs of solutions and the type of equation (II.6) holds only for elliptic solutions of equation (II.6). The elliptic solutions of this equation are necessarily convex or concave functions. It is sufficient to consider only convex solutions of equation (II.6). If equation (II.6) has convex solutions, then the function D takes only positive values.

In Part II we are concerned with investigations of the boundary value problems for weak and generalized solutions of the Monge–Ampere equations (II.6). These solutions are general convex and concave functions. *The geometric theory of Monge–Ampere equations* is just the union of the concepts, techniques and results of these investigations. Just as in other branches of modern mathematics, the study of non-smooth objects, which are in the present case weak and generalized solutions of equation (II.6), is not the main goal. We present them just for deeper understanding of smooth subjects. The theory of weak and generalized solutions of various boundary value problems for equation (II.6) is an excellent illustration of this statement, because the proofs of all existence theorems for weak and generalized solutions of such boundary value problems are based on simple principles of compactness in appropriate function spaces. Moreover these theorems are proved under more general and natural conditions than the corresponding theorems for classical solutions.

Since weak and generalized solutions of equation (II.6) are in the set of all general convex and concave functions, the question of C^m-smoothness, $m \geq 2$, of such solutions arises. This question should be considered in two ways. Let G be the domain of equation (II.6). First of all we want to find conditions on the function $D(x,u,p)$, which provide the C^m-smoothness of weak and generalized solutions of equation (II.6) in G. The second problem is to find conditions on the function $D(x,u,p)$ and the boundary data, which provide the C^m-smoothness of the same solutions in $\overline{G} = G \cup \partial G$. The positive solution of these problems opens the way for the application of the techniques and results of the geometric theory of Monge–Ampere operators and equations to classical problems in PDE and Differential Geometry. We consider the C^m-smoothness of solutions for equation (II.6) in Chapter 6.