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Preface

This book is intended to be a detailed and carefully written account of various versions of the Littlewood-Paley theorem and of some of its applications, together with indications of its general significance in Fourier multiplier theory. We have striven to make the presentation self-contained and unified, and adapted primarily for use by graduate students and established mathematicians who wish to begin studies in these areas: it is certainly not intended for experts in the subject.

It has been our experience, and the experience of many of our students and colleagues, that this is an area poorly served by existing books. Their accounts of the subject tend to be either ill-suited to the needs of a beginner, or fragmentary, or, in one or two instances, obscure. We hope that our book will go some way towards filling this gap in the literature.

Our presentation of the Littlewood-Paley theorem proceeds along two main lines, the first relating to singular integrals on locally compact groups, and the second to martingales. Both classical and modern versions of the theorem are dealt with, appropriate to the classical groups $\mathbb{R}^n$, $\mathbb{Z}^n$, $\mathbb{T}^n$ and to certain classes of disconnected groups. It is for the disconnected groups of Chapters 4 and 5 that we give two separate accounts of the Littlewood-Paley theorem: the first Fourier analytic, and the second probabilistic.

Some central results about multipliers of $\mathfrak{F}L^p(1 < p < \infty)$ are established, either collaterally with the Littlewood-Paley theorem, or as deductions from that theorem; for instance the famous theorems of M. Riesz, Marcinkiewicz, and Stečkin. In proving these concrete results, we have had also to develop or use certain portions of the *general* theory of Fourier multipliers. We think that the mix thus produced is a healthy one and that our book can therefore serve as a balanced introduction to the study of Fourier multipliers of L^p on LCA groups.

The applications, in the last chapter, to lacunary sets and Fourier multiplier theory, are meant to illustrate the importance of the Littlewood-Paley theorem as a tool in harmonic analysis. This is an idea which has been exploited with considerable success in recent years.

In addition to the general developments and applications just mentioned, our book contains a few results which, as far as we know, are new.

There are places where some readers may accuse us of pedantry. The fact is that we have merely tried to provide some details—possibly routine for the expert but troublesome for some others—which are almost always brushed aside with something close to contempt. The stock instance is the distinction between functions and the corresponding function-classes modulo negligible functions. Very often the vagueness and the cure are apparent even to a beginner. This is not always the case, however, and in such instances we have tried to replace the familiar hand-waving by something a little more convincing.

We have, in the main text, deliberately ignored historical and bibliographical matters. This is because we wished to pursue the mathematics, without undue distraction, to the goals we had set ourselves. Since, however, some bibliographical indications of the original sources of the main theorems are desirable, we have added a few comments of the kind in the Historical Notes at the end of the book. While we hope these Notes will be useful to some of our readers, we want to make it plain that they should be regarded as no more than a rough and incomplete guide to the literature.

We are indebted to many friends for encouragement and assistance in this enterprise.

Through his collaboration with the second of us, Alessandro Figà-Talamanca contributed indirectly in many ways to the present book. Even though it would be impossible to specify precise instances where his outlook, enthusiasm and ideas have made themselves felt in our writing, we (especially G^2) are well aware of his influence and are pleased to acknowledge it.

Edwin Hewitt has served on numerous occasions as a source of encouragement. It is due in no small measure to his good influence on us that our early plans for the book have now come to fruition.

We appreciate the exceedingly generous assistance of Jeff Sanders with the proof reading. Warm thanks are also due to Jan May for her expert typing of an early draft of our work, and to Cheryl Vertigan for producing the beautifully typed final version and for help with marking it up for the printers.

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