



J. W. S. Cassels (known to his friends by the Gaelic form “Ian” of his first name) was born of mixed English-Scottish parentage on 11 July 1922 in the picturesque cathedral city of Durham. With a first degree from Edinburgh, he commenced research in Cambridge in 1946 under L. J. Mordell, who had just succeeded G. H. Hardy in the Sadleirian Chair of Pure Mathematics. He obtained his doctorate and was elected a Fellow of Trinity College in 1949. After a year in Manchester, he returned to Cambridge and in 1967 became Sadleirian Professor. He was Head of the Department of Pure Mathematics and Mathematical Statistics from 1969 until he retired in 1984.

Cassels has contributed to several areas of number theory and written a number of other expository books:

- *An introduction to diophantine approximations*
- *Rational quadratic forms*
- *Economics for mathematicians*
- *Local fields*
- *Lectures on elliptic curves*
- *Prolegomena to a middlebrow arithmetic of curves of genus 2* (with E. V. Flynn).

Classics in Mathematics

J.W.S. Cassels An Introduction to the Geometry of Numbers

Springer

Berlin

Heidelberg

New York

Barcelona

Budapest

Hong Kong

London

Milan

Paris

Santa Clara

Singapore

Tokyo

J.W.S. Cassels

An Introduction to the Geometry of Numbers

Reprint of the 1971 Edition



Springer

J.W.S. Cassels
University of Cambridge
Department of Pure Mathematics
and Mathematical Statistics
16, Mill Lane
CB2 1SB Cambridge
United Kingdom

Originally published as Vol. 99 of the
Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen

Mathematics Subject Classification (1991): 10Exx

CIP data applied for

Die Deutsche Bibliothek – CIP-Einheitsaufnahme

Cassels, John W.S.:

An introduction to the geometry of numbers / J.W.S. Cassels - Reprint of the 1971 ed. - Berlin; Heidelberg; New York; Barcelona; Budapest; Hong Kong; London; Milan; Paris; Santa Clara; Singapore; Tokyo: Springer, 1997

(Classics in mathematics)

ISBN-13: 978-3-540-61788-4

e-ISBN-13: 978-3-642-62035-5

DOI: 10.1007/978-3-642-62035-5

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilm or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1997

The use of general descriptive names, registered names, trademarks etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

SPIN 10554506

41/3143-5 4 3 2 1 0 – Printed on acid-free paper

J. W. S. Cassels

An Introduction to the Geometry of Numbers

Second Printing, Corrected



Springer-Verlag Berlin · Heidelberg · New York 1971

Prof. Dr. J. W. S. Cassels
Professor of Mathematics, University of Cambridge, G. B.

Geschäftsführende Herausgeber:

Prof. Dr. B. Eckmann
Eidgenössische Technische Hochschule Zürich

Prof. Dr. B. L. van der Waerden
Mathematisches Institut der Universität Zürich

AMS Subject Classifications (1970): 10 E xx

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically those of translation, reprinting, re-use of illustrations, broadcasting, reproduction by photocopying machine or similar means, and storage in data banks.

Under § 54 of the German Copyright Law where copies are made for other than private use, a fee is payable to the publisher, the amount of the fee to be determined by agreement with the publisher.

© by Springer-Verlag Berlin · Heidelberg 1959, 1971. Library of Congress Catalog Card Number 75-154801.

Preface

Of making many bookes there is no end, and much studie is a wearinesse of the flesh.

Ecclesiastes XII, 12.

When I first took an interest in the Geometry of Numbers, I was struck by the absence of any book which gave the essential skeleton of the subject as it was known to the experienced workers in the subject. Since then the subject has developed, as will be clear from the dates of the papers cited in the bibliography, but the need for a book remains. This is an attempt to fill the gap. It aspires to acquaint the reader with the main lines of development, so that he may with ease and pleasure follow up the things which interest him in the periodical literature. I have attempted to make the account as self-contained as possible.

References are usually given to the more recent papers dealing with a particular topic, or to those with a good bibliography. They are given only to enable the reader to amplify the account in the text and are not intended to give a historical picture. To give anything like a reasonable account of the history of the subject would have involved much additional research.

I owe a particular debt of gratitude to Professor L. J. MORDELL, who first introduced me to the Geometry of Numbers.

The proof-sheets have been read by Professors K. MAHLER, L. J. MORDELL and C. A. ROGERS. It is a pleasure to acknowledge their valuable help and advice both in detecting errors and obscurities and in suggesting improvements. Dr. V. ENNOLA has drawn my attention to several slips which survived into the second proofs.

I should also like to take the opportunity to thank Professor F. K. SCHMIDT and the Springer-Verlag for accepting this book for their celebrated yellow series and the Springer-Verlag for its readiness to meet my typographical whims.

Cambridge, June, 1959

J. W. S. CASSELS

Contents

	Page
Notation	VIII
Prologue	1
Chapter I. Lattices	9
1. Introduction	9
2. Bases and sublattices	9
3. Lattices under linear transformation	19
4. Forms and lattices	20
5. The polar lattice	23
Chapter II. Reduction	26
1. Introduction	26
2. The basic process	27
3. Definite quadratic forms	30
4. Indefinite quadratic forms	35
5. Binary cubic forms	51
6. Other forms	60
Chapter III. Theorems of BLICHFELDT and MINKOWSKI	64
1. Introduction	64
2. BLICHFELDT's and MINKOWSKI's theorems	68
3. Generalisations to non-negative functions	73
4. Characterisation of lattices	78
5. Lattice constants	80
6. A method of MORDELL	84
7. Representation of integers by quadratic forms	98
Chapter IV. Distance functions	103
1. Introduction	103
2. General distance-functions	105
3. Convex sets	108
4. Distance functions and lattices	119
Chapter V. MAHLER's compactness theorem	121
1. Introduction	121
2. Linear transformations	122
3. Convergence of lattices	126
4. Compactness for lattices	134
5. Critical lattices	141
6. Bounded star-bodies	145
7. Reducibility	152
8. Convex bodies	155
9. Spheres	163
10. Applications to diophantine approximation	165
Chapter VI. The theorem of MINKOWSKI-HLAWKA	175
1. Introduction	175
2. Sublattices of prime index	178

	Page
3. The Minkowski-Hlawka theorem	181
4. SCHMIDT's theorems	184
5. A conjecture of ROGERS	187
6. Unbounded star-bodies	189
Chapter VII. The quotient space	194
1. Introduction	194
2. General properties	194
3. The sum theorem	198
Chapter VIII. Successive minima	201
1. Introduction	201
2. Spheres	205
3. General distance-functions	207
4. Convex sets	213
5. Polar convex bodies	219
Chapter IX. Packings	223
1. Introduction	223
2. Sets with $V(\mathcal{S}) = 2^n \Delta(\mathcal{S})$	228
3. VORONOI's results	231
4. Preparatory lemmas	235
5. FEJES TÓTH's theorem	240
6. Cylinders	245
7. Packing of spheres	246
8. The product of n linear forms	250
Chapter X. Automorphs	256
1. Introduction	256
2. Special forms	266
3. A method of MORDELL	268
4. Existence of automorphs	279
5. Isolation theorems	286
6. Applications of isolation	295
7. An infinity of solutions	298
8. Local methods	301
Chapter XI. Inhomogeneous problems	303
1. Introduction	303
2. Convex sets	309
3. Transference theorems for convex sets	313
4. The product of n linear forms	322
Appendix	332
References	334
Index	343

Notation

An effort has been made to distinguish different types of mathematical object by the use of different alphabets. It is not necessary to describe the scheme in full since an acquaintance with it is not presupposed. However the following conventions are made throughout the book without explicit mention.

Bold Latin letters (large and small) always denote vectors. The dimensions is n , unless the contrary is explicitly stated: and the letter n is not used otherwise, except in one or two places where there can be no fear of ambiguity. The co-ordinates of a vector are denoted by the corresponding italic letter with a suffix $1, 2, \dots, n$. If the bold letter denoting the vector already has a suffix, then that is put after the co-ordinate suffix. Thus:

$$\begin{aligned}\mathbf{a} &= (a_1, \dots, a_n) \\ \mathbf{b}_r &= (b_{1r}, \dots, b_{nr}) \\ \mathbf{X}'_\varepsilon &= (X'_{1\varepsilon}, \dots, X'_{n\varepsilon}).\end{aligned}$$

The origin is always denoted by \mathbf{o} . The length of \mathbf{x} is

$$|\mathbf{x}| = (x_1^2 + \dots + x_n^2)^{\frac{1}{2}}.$$

Sanserif Greek capitals, in particular Λ, M, N, Γ , denote lattices.

The notation $d(\Lambda)$, $\Delta(\mathcal{S})$, $V(\mathcal{S})$ for respectively the determinant of the lattice Λ and for the lattice-constant and volume of a set \mathcal{S} will be standard, once the corresponding concepts have been introduced.

Chapters are divided into sections with titles. These sections are subdivided, for convenience, into subsections, which are indicated by a decimal notation. The numbering of displayed formulae starts afresh in each subsection. The prologue is just subdivided into sections without titles, and it was convenient to number the displayed formulae consecutively throughout.