George Pólya • Gabor Szegő

# Problems and Theorems in Analysis II

## Theory of Functions. Zeros. Polynomials. Determinants. Number Theory. Geometry

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George Pólya †
Gabor Szegő †

*Translator:*
C. E. Billigheimer
McMaster University
Mathematics
Hamilton, Ontario
Canada

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G. Pólya G. Szegö

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Volume II

Theory of Functions · Zeros · Polynomials
Determinants · Number Theory · Geometry

Translation by C. E. Billigheimer

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George Pólya . Gabor Szegö
Stanford University, Stanford, California, USA

Claude Elias Billigheimer
McMaster University, Hamilton, Ontario, Canada
and Maimonides College, Toronto, Ontario, Canada

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# Notation and Abbreviations

We have attempted to be as consistent as possible in regard to notation and abbreviations and to denote quantities of the same nature by the same symbol, at least within the same part. A particular notation may be specified for a few sections. Otherwise the meaning of every letter is explained anew in every problem except when we refer to a previous problem. A problem that is closely related to the preceding one is introduced by the remark "continued"; if it is related to some other problem the relevant number is mentioned, e.g. "continuation of **136**".

We denote parts by roman numerals, chapters (where necessary) by arabic numerals. The problems are numbered in bold-face. Within the same part only the number of the problem is given; if, however, we refer to another part its number is also indicated. For example if we refer to problem (or solution) **123** of Part IV in a problem (or solution) of Part IV we write "**123**"; if we refer to it in a problem (or solution) of any other part we write "IV **123**".

Remarks in square brackets [ ] in a problem are hints, while in a solution (particularly at the beginning of the solution) they are citations or references to other problems that are used in various steps of the proof. All other remarks appear in ordinary parentheses. A reference to a problem number indicates in general that one should consult both problem and solution, unless the opposite is explicitly stated, e.g. "solution **75**".

Almost always references to the sources are given only in the solution. If a problem has already appeared in print, this fact is indicated in the citations. If the author but no bibliography is mentioned, the problem has been communicated to us as a new problem. Problems whose number is preceded by the symbol * (as in ***206** of Part IV) or contains a decimal point (as in **174.1** of Part IV) are new, that is they are either not contained in the original German edition, or else are contained there but are essentially modified in the present English version. If the problem is the same as in the original edition but the solution has some essentially new feature, the symbol * is used only in the solution. The abbreviations of the names of journals are taken from the index of Mathematical Reviews and, if not listed there, from the World List of Scientific Periodicals Published 1900–1960, Peter Brown, British Museum, Washington, Butterworths, 1963.

The most frequently quoted journals are:

| | |
|---|---|
| Abh. Akad. Wiss. St. Petersburg | = Akademie der Wissenschaften, St. Petersburg |
| Acta Math. | = Acta Mathematica, Stockholm |
| Acta Soc. Sc. Fennicae | = Acta Societatis Scientiae Fennicae |
| Amer. Math. Monthly | = American Mathematician Monthly |
| Arch. Math. Phys. | = Archiv der Mathematik und Physik |
| Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. | = Atti dell' Accademia Nazionale dei Lincei Rendiconti. Classe di Scienze Fisiche, Matematiche e Naturali, Roma |
| Berlin. Ber. | = Berliner Berichte |
| C.R. Acad. Sci. (Paris) Ser. A–B | = Comptes rendus hebdomadaires des séances de l'Académie des Sciences, Paris, Séries A et B |
| Giorn. Mat. Battaglini | = Giornale di Matematiche di Battaglini |
| Jber. deutsch. Math. Verein. | = Jahresbericht der deutschen Mathematiker-Vereinigung |
| J. Math. spéc. | = Journal de Mathématiques Spécieles, Paris |
| J. reine angew. Math. | = Journal für die reine und angewandte Mathematik |
| Math. Ann. | = Mathematische Annalen |
| Math. és term ért | = Matematikai és természettudományi értesitö |
| Math. Z. | = Mathematische Zeitschrift |
| Münchner Ber. | = Münchner Berichte |

| | |
|---|---|
| Nachr. Akad. Wiss. Göttingen | = Nachrichten der Gesellschaft der Wissenschaften Göttingen |
| Nouv. Annls Math. | = Nouvelles Annales de mathématiques |
| Nyt. Tidsskr. | = Nyt tidsskrift for matematik |
| Proc. Amer. Math. Soc. | = Proceedings of the American Mathematical Society |
| Proc. Lond. Math. Soc. | = Proceedings of the London Mathematical Society |
| Trans. Amer. Math. Soc. | = Transactions of the American Mathematical Society |

The following textbooks are quoted repeatedly and are usually cited by the name of the author only or by a suitable abbreviation (e.g. Hurwitz-Courant; MPR.):

G.H. Hardy and E.M. Wright: An Introduction to the Theory of Numbers, 4th Ed. Oxford: Oxford University Press 1960.
E. Hecke: Vorlesungen über die Theorie der algebraischen Zahlen, New York: Chelsea Publishing Co. 1948.
E. Hille: Analytic Function Theory, Vol. I: Boston – New York – Chicago – Atlanta – Dallas – Palo Alto – Toronto – London: Ginn & Co. 1959; Vol. II: Waltham/Mass. – Toronto – London: Blaisdell Publishing Co. 1962.
A. Hurwitz – R. Courant: Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen, 4th Ed. Berlin – Göttingen – Heidelberg – New York: Springer 1964.
K. Knopp: Theory and Applications of Infinite Series, 2nd Ed. London – Glasgow: Blackie & Son 1964.
G. Kowalewski: Einführung in die Determinantentheorie, 4th Ed. Berlin: Walter de Gruyter 1954.
G. Pólya: How to Solve It, 2nd Ed. Princeton: Princeton University Press 1971. Quoted as HSI.
G. Pólya: Mathematics and Plausible Reasoning, Vols. 1 and 2, 2nd Ed. Princeton: Princeton University Press 1968. Quoted as MPR.
G. Pólya: Mathematical Discovery, Vols. 1 and 2, Cor. Ed. New York: John Wiley & Sons 1968. Quoted as MD.
G. Szegö: Orthogonal Polynomials, American Mathematical Society Colloquium Publications Vol. XXIII, 3rd Ed. New York: American Mathematical Society 1967.
E.C. Titchmarsh: The Theory of Functions, 2nd Ed. Oxford – London – Glasgow – New York – Melbourne – Toronto: Oxford University Press 1939.
E.T. Whittaker and G.N. Watson: A Course of Modern Analysis, 4th Ed. London: Cambridge University Press 1952.

The following notation and abbreviations are used throughout the book:
$a_n \to a$ means "$a_n$ tends to $a$ as $n \to \infty$."

$a_n \sim b_n$ (read "$a_n$ is asymptotically equal to $b_n$") means "$b_n \neq 0$ for sufficiently large $n$ and $\frac{a_n}{b_n} \to 1$ as $n \to \infty$".

$O(a_n)$, with $a_n > 0$, denotes a quantity that, divided by $a_n$, remains bounded, $o(a_n)$ a quantity that, divided by $a_n$, tends to 0 as $n \to \infty$.

Such notation is used analogously in limit processes other than for $n \to \infty$.

$x \to a+0$ means "$x$ converges to $a$ from the right";

$x \to a-0$ means "$x$ converges to $a$ from the left".

$\exp(x) = e^x$, where $e$ is the base of natural logarithms.

Given $n$ real numbers $a_1, a_2, \ldots, a_n$, $\max(a_1, a_2, \ldots, a_n)$ denotes the largest (or one of the largest) and $\min(a_1, a_2, \ldots, a_n)$ the smallest (or one of the smallest) of the numbers $a_1, a_2, \ldots, a_n$. Max $f(x)$ and min $f(x)$ have an analogous meaning for a real function defined on an interval $a, b$, provided $f(x)$ assumes a maximum or a minimum on $a, b$. Otherwise we retain the same notation for the least upper bound and the greatest lower bound, respectively. Analogous notation is used in the case of functions of a complex variable.

sgn $x$ denotes the signum (Kronecker) function:

$$\operatorname{sgn} x = \begin{cases} +1 & \text{for } x > 0 \\ 0 & \text{for } x = 0 \\ -1 & \text{for } x < 0. \end{cases}$$

$[x]$ denotes the greatest integer that is not greater than $x$ $(x-1<[x]\leqq x)$. Square brackets are, however, also used instead of ordinary parentheses where there is no danger of confusion. (They are also used in a very special sense restricted to Part I, Chap. 1, § 5.)

$\bar{z}$ is the conjugate to the complex number $z$.

For the determinant with general term $a_{\lambda,\mu}$, $\lambda$, $\mu=1, 2, \ldots, n$, we use the abbreviated notation

$$|a_{\lambda\mu}|_1^n \quad \text{or} \quad |a_{\lambda\mu}|_{\lambda,\mu=1,2,\ldots,n} \quad \text{or} \quad |a_{\lambda 1}, a_{\lambda 2}, \ldots, a_{\lambda n}|_1^n.$$

A non-empty connected open set (containing only interior points) is called a *region*. The closure of a region (the union of the open set and of its boundary) is called a *domain*. As this terminology is not the one most frequently used, we shall sometimes emphasize it by speaking of an "open region" and a "closed domain".

A *continuous curve* is defined as a single-valued continuous image of the interval $0\leqq t\leqq 1$, i.e. as the set of points $z=x+iy$, where $x=\varphi(t)$, $y=\psi(t)$, with $\varphi(t)$ and $\psi(t)$ both continuous functions on the interval $0\leqq t\leqq 1$. The curve is *closed* if $\varphi(0)=\varphi(1)$, $\psi(0)=\psi(1)$, and is *without double points* if $\varphi(t_1)=\varphi(t_2)$, $\psi(t_1)=\psi(t_2)$, $t_1<t_2$, imply $t_1=0$, $t_2=1$. A curve without double points is also called a *simple curve*. A simple, continuous curve that is not closed is often referred to as a *simple arc*.

A simple closed continuous curve (a *Jordan* curve) in the plane determines two regions of which it forms the common boundary.

The paths of integration of line integrals or complex integrals are assumed to be continuous and rectifiable.

$(a, b)$ denotes the open interval $a<x<b$, $[a, b)$ the half-open interval $a\leqq x<b$, $(a, b]$ the half-open interval $a<x\leqq b$, $[a, b]$ the closed interval $a\leqq x\leqq b$. When we do not need to distinguish between these four cases we use the term "interval $a$, $b$".

"Iff" is used occasionally as an abbreviation for "if and only if".