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Geometries and Groups

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by M. Reid

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Preface

This book is devoted to the theory of geometries which are locally Euclidean, in the sense that in small regions they are identical to the geometry of the Euclidean plane or Euclidean 3-space. Starting from the simplest examples, we proceed to develop a general theory of such geometries, based on their relation with discrete groups of motions of the Euclidean plane or 3-space; we also consider the relation between discrete groups of motions and crystallography. The description of locally Euclidean geometries of one type shows that these geometries are themselves naturally represented as the points of a new geometry. The systematic study of this new geometry leads us to 2-dimensional Lobachevsky geometry (also called non-Euclidean or hyperbolic geometry) which, following the logic of our study, is constructed starting from the properties of its group of motions. Thus in this book we would like to introduce the reader to a theory of geometries which are different from the usual Euclidean geometry of the plane and 3-space, in terms of examples which are accessible to a concrete and intuitive study. The basic method of study is the use of groups of motions, both discrete groups and the groups of motions of geometries.

The book does not presuppose on the part of the reader any preliminary knowledge outside the limits of a school geometry course. We have in mind a wide circle of possible readers: students of mathematics and physics in universities and technical colleges, high school teachers and students in the upper classes of high school... We hope that reading this book will enable even a reader with no professional mathematical training to become acquainted with one of the most attractive aspects of mathematics: that many of its problems are solved using methods and concepts which at first sight have nothing whatever to do with the original problem. It is only after fairly lengthy development that these methods lead to a solution of the problem which led to their appearance, and they often open up before the investigator a completely new field of study. This internal development of mathematics, whereby the needs of one area lead to the creation of new areas of research is complemented by the extraordinary phenomenon of its unity: theories created for various ends and developing in different directions unexpectedly turn out to be closely related. Naturally, in order to get a feeling for these special features of mathematical research, the reader must be ready to spend both time and effort overcoming the difficulties which he may face in reading the book; the difficulties are not caused by the use of complex techniques - the reader will be able to get by with a school geometry course - but by the need to get used to more complex and longer mathematical arguments.

The book divides into four chapters. It is our hope that the first can be read by a high school student with interest in mathematics. In it we treat the basic examples, which will allow the reader to gain some geometrical intuition in the new subject. Reading this chapter only will already provide some first impressions of the subject matter of the book.

The second chapter is the heart of the book: in it we introduce the new method, with which the problem stated at the end of Chapter I can be solved. This chapter is distinctly harder than the first; it contains the proofs of a number of theorems, some of which are not so easy even for a professional mathematician. It is, however, our hope that while working on Chapter I, the reader will have acquired certain abilities which will help him to overcome the difficulties of Chapter II. The reader who has worked his way through the first two chapters will already have a complete impression of the theory which forms the subject of the book. The remainder of the book relates this theory to other questions.

The reader will probably find Chapter III rather easier. Its first section §11 generalises the theory, constructed so far in two dimensions, to the 3-dimensional case; the second section §12 deals with the relation of this theory to crystallography. In the final fourth chapter of the book the logic of the development of our theory leads naturally to a completely new fundamental concept of geometry, Lobachevsky geometry.

Almost each section ends with some exercises. They do not aim to introduce the reader to new facts; in the main, they are intended to help him check the extent to which he has grasped the preceding text. For this reason they are as a rule very simple.

Some references for further reading are included in the text. The reader who would like to pursue the questions treated in this book in more depth is recommended the following books:

E. Cartan, Geometry of Riemannian spaces, Gauthier-Villars, Paris, 1928, 2nd ed., 1951 reprinting and Math. Sci. Press, Brookline, 1983;

J.A. Wolf, Spaces of constant curvature, McGraw-Hill, New York, 1967;

S. Helgason, Differential geometry and symmetric spaces, Academic Press, New York, 1967;

L.R. Ford, Automorphic functions, Chelsea, New York 1951;

M. Klemm, Symmetrien von Ornamenten und Kristallen, Springer, New York - Berlin, 1982;

together with the survey article:

P. Scott, The geometries of 3-manifolds, Bulletin of London Math. Soc. **15** (1983), p. 401-487.

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