

Universitext

Springer-Verlag Berlin Heidelberg GmbH

Vladimir Boltyanski
Horst Martini
Petru S. Soltan

Excursions into Combinatorial Geometry

With 264 Figures



Springer

Vladimir Boltyanski
Steklov Mathematical Institute
Vavilov Street 42
117966 Moscow, Russia

Horst Martini
Faculty of Mathematics
TU Chemnitz-Zwickau
PSF 964
09009 Chemnitz, Germany

Petru S. Soltan
Faculty of Mathematics
Moldavian State University
Street Mateevici 60
Kishinev, Moldova

The cover picture shows a Borsuk partition of the known polyhedral cover constructed by Branko Grünbaum

Cataloging-in-Publication applied for

Die Deutsche Bibliothek - CIP-Einheitsaufnahme

Boltjanskij, Vladimir G.:
Excursion into combinatorial geometry / Vladimir Boltyanski ;
Horst Martini ; Petru S. Soltan. - Berlin ; Heidelberg ; New
York ; Barcelona ; Budapest ; Hong Kong ; London ; Milan ;
Paris ; Santa Clara ; Singapore ; Tokyo : Springer, 1997
(Universitext)
ISBN 978-3-540-61341-1 ISBN 978-3-642-59237-9 (eBook)
DOI 10.1007/978-3-642-59237-9
NE: Martini, Horst.; Soltan, Petru S.:

Mathematics Subject Classification (1991) 52-01, 52-02, 52A30, 52A35, 52A37, 52C17

ISBN 978-3-540-61341-1

This work is subject to copyright. All rights are reserved, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other way, and storage in data banks. Duplication of this publication or parts thereof is permitted only under the provisions of the German Copyright Law of September 9, 1965, in its current version, and permission for use must always be obtained from Springer-Verlag. Violations are liable for prosecution under the German Copyright Law.

© Springer-Verlag Berlin Heidelberg 1997
Originally published by Springer-Verlag Berlin Heidelberg New York in 1997

Typesetting: Editing and reformatting of the authors' input files by Springer-Verlag
SPIN: 10521765 41/3143-543210 - Printed on acid-free paper.

To our wives

Lilia, Angelika and Luba

with love and thanks

Preface

Geometry undoubtedly plays a central role in modern mathematics. And it is not only a physiological fact that 80 % of the information obtained by a human is absorbed through the eyes. It is easier to grasp mathematical concepts and ideas visually than merely to read written symbols and formulae. Without a clear geometric perception of an analytical mathematical problem our intuitive understanding is restricted, while a geometric interpretation points us towards ways of investigation.

Minkowski's convexity theory (including support functions, mixed volumes, finite-dimensional normed spaces etc.) was considered by several mathematicians to be an excellent and elegant, but useless mathematical device. Nearly a century later, geometric convexity became one of the major tools of modern applied mathematics. Researchers in functional analysis, mathematical economics, optimization, game theory and many other branches of our field try to gain a clear geometric idea, before they start to work with formulae, integrals, inequalities and so on. For examples in this direction, we refer to [Mal] and [B-M 2].

Combinatorial geometry emerged this century. Its major lines of investigation, results and methods were developed in the last decades, based on seminal contributions by O. Helly, K. Borsuk, P. Erdős, H. Hadwiger, L. Fejes Tóth, V. Klee, B. Grünbaum and many other excellent mathematicians. Many of the open questions in this field can be explained in simple terms, even to non-mathematicians. But the solutions to these questions are often so complicated that every problem resolved is a major event.

It is impossible to give a complete list of references to this area, but the following books and surveys cover the major developments in combinatorial geometry: [D-G-K], [Grü 3,7,8], [H-D-K], [B-G 1,2], [B-SP 3], [Ec], [Rea], [Si], [SV 4], [B-So], chapters D and E in [C-F-G], [Be 3], [E], [Schm], [G-G-L], and [P-A]. We view the present book as a natural continuation of [B-G 1] and [B-SP 3]. The wider field of geometric convexity is covered for the most part in [Bl 2], [Eg 2,3], [Kl 2], [Ha 5], [Grü 6], [T-W], [G-W 1,2], [Schm], [B-M-S-W], [Zi], [Gar], [Tho] and several other publications.

Our book contains a collection of characteristic problems, tied together by several concepts: d -convexity, H -convexity, generalizations of the classical Helly theorem and its relatives, the Helly dimension of convex bodies, different covering and illumination problems etc. In addition, we introduce a

new class of convex bodies, namely the family of belt bodies. The notion of a “belt body” is a natural generalization of the notion “zonoid”. The belt bodies are dense in the set of all compact, convex bodies, whereas the zonoids are not even dense in the set of centrally symmetric, compact, convex bodies. It is shown that some difficult questions from combinatorial geometry can be completely resolved for belt bodies.

Our book can be used for different purposes. First, it is suitable for advanced undergraduate and graduate classes. With this audience in mind, we have amply illustrated the material with figures and provided numerous exercises. We hope readers will find this book stimulating to their geometric insight. Second, the book may serve as a reference to the current state of the art in some fields belonging to combinatorial geometry. Finally, specialists in these fields will find a collection of open problems in the last chapter.

A diagram at the beginning of the book illustrates the interdependence of the sections, and in addition some special “excursions” are described by section numbers. Note furthermore that some sections contain passages of survey character, that may be of less relevance, in first reading, to students. This applies in particular to sections 31 and 34.

The authors are grateful to I. Gohberg, B. Grünbaum, P. M. Gruber, J. Kincses, V. Klee, E. Makai Jr., and B. Weissbach for many stimulating discussions. They are also indebted to Mrs. Diana Lange for typing the manuscript in LATEX.

In addition, the authors are indebted to “Deutsche Forschungsgemeinschaft” for having supported longer visits of the first named author and the third named author at the Technical University Chemnitz-Zwickau.

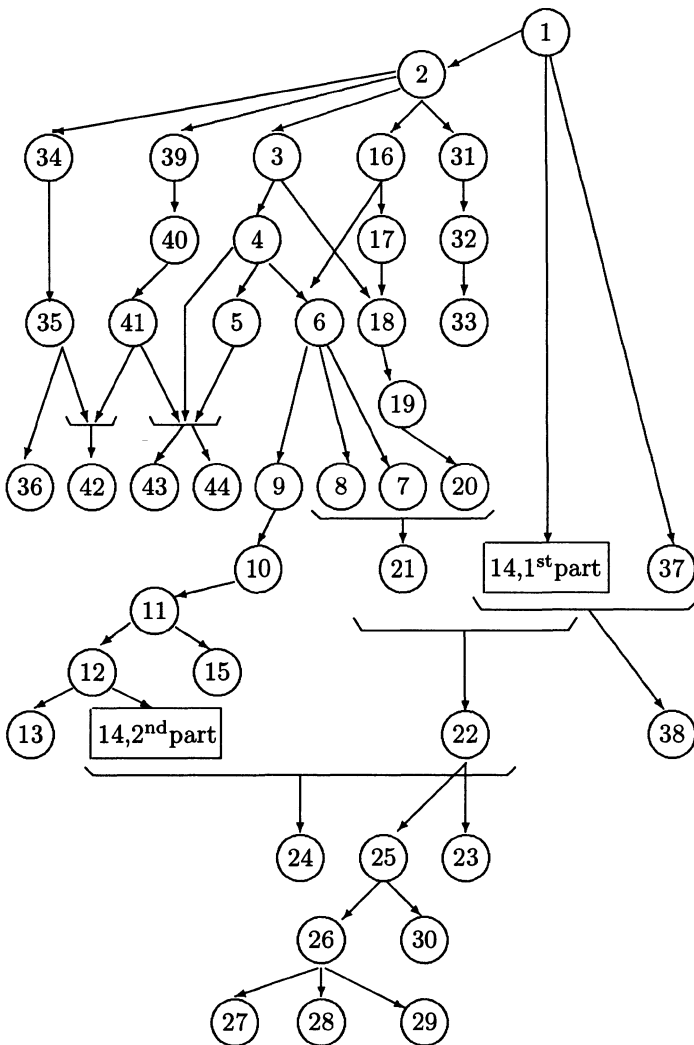
V. Boltyanski, H. Martini, and P.S. Soltan

Table of Contents

I.	Convexity	1
	§1 Convex sets	1
	§2 Faces and supporting hyperplanes	6
	§3 Polarity	12
	§4 Direct sum decompositions	15
	§5 The lower semicontinuity of the operator “exp”	20
	§6 Convex cones	27
	§7 The Farkas Lemma and its generalization	36
	§8 Separable systems of convex cones	41
II.	<i>d</i>-Convexity in normed spaces	49
	§9 The definition of <i>d</i> -convex sets	49
	§10 Support properties of <i>d</i> -convex sets	58
	§11 Properties of <i>d</i> -convex flats	65
	§12 The join of normed spaces	74
	§13 Separability of <i>d</i> -convex sets	81
	§14 The Helly dimension of a set family	91
	§15 <i>d</i> -Star-shaped sets	99
III.	<i>H</i>-convexity	109
	§16 The functional md for vector systems	109
	§17 The ε -displacement Theorem	115
	§18 Lower semicontinuity of the functional md	121
	§19 The definition of <i>H</i> -convex sets	125
	§20 Upper semicontinuity of the <i>H</i> -convex hull	127
	§21 Supporting cones of <i>H</i> -convex bodies	135
	§22 The Helly Theorem for <i>H</i> -convex sets	143
	§23 Some applications of <i>H</i> -convexity	149
	§24 Some remarks on connection between <i>d</i> -convexity and <i>H</i> -convexity	154
IV.	The Szökefalvi-Nagy Problem	163
	§25 The Theorem of Szökefalvi-Nagy and its generalization	163

§26	Description of vector systems with $\text{md } H = 2$ that are not one-sided	173
§27	The 2-systems without particular vectors	177
§28	The 2-system with particular vectors	184
§29	The compact, convex bodies with $\text{md } M = 2$	188
§30	Centrally symmetric bodies	198
V.	Borsuk's partition problem	209
§31	Formulation of the problem and a survey of results	209
§32	Bodies of constant width in Euclidean and normed spaces ..	227
§33	Borsuk's problem in normed spaces	239
VI.	Homothetic covering and illumination	255
§34	The main problem and a survey of results	255
§35	The hypothesis of Gohberg-Markus-Hadwiger	275
§36	The infinite values of the functionals b, b', c, c'	288
§37	Inner illumination of convex bodies	301
§38	Estimates for the value of the functional $p(K)$	309
VII.	Combinatorial geometry of belt bodies	319
§39	The integral representation of zonoids	319
§40	Belt vectors of a compact, convex body	327
§41	Definition of belt bodies	333
§42	Solution of the illumination problem for belt bodies	339
§43	Solution of the Szökefalvi-Nagy problem for belt bodies ...	346
§44	Minimal fixing systems	352
VIII.	Some research problems	365
	Bibliography	393
	Author Index	411
	Subject Index	415
	List of Symbols	419

Dependence of sections



Some excursions

d-convexity:

Sections 1, 2, 3, 4, 6, 9, 10, 11, 12, 13, 14, 15
(and the section 24 after studying *H*-convexity).

H-convexity:

Sections 1, 2, 3, 4, 5, 6, 7, 8, 14 (1st part), 16, 17, 18, 19, 20, 21, 22, 23
(and the section 24 after studying *d*-convexity).

Szőkefalvi-Nagy problem:

Sections 1, 2, 3, 4, 5, 6, 7, 8, 14 (1st part), 16, 17, 18, 19, 20, 21, 22, 25,
26, 27, 28, 29, 30 (and, in addition, sections 39, 40, 41, 43).

Borsuk problem:

Sections 1, 2, 31, 32, 33.

Gohberg-Markus-Hadwiger problem:

Sections 1, 2, 34, 35, 36
(and, in addition, sections 39, 40, 41, 42).

Inner illumination problem:

Sections 1, 2, 14 (1st part), 37, 38.

Minimal fixing systems:

Sections 1, 2, 3, 4, 16, 39, 40, 41, 44.