Part V

Artificial Boundary Conditions for Stationary Problems

A typical example of a problem requiring the construction of artificial boundary conditions, is the problem of calculating the velocities and pressure of the air flow around a body, usually in the close vicinity of the body. However, for computation we have to take a considerably larger neighborhood.

The point is that on the boundary separating the computational domain we must construct artificial boundary conditions for completing the statement of the problem. These artificial boundary conditions must exactly replace the equations for the other part of the space as well as the conditions at infinity. The enlarged dimensions of the computational domain, as compared to the neighborhood of the body that is of immediate interest, are justified by the fact that it is easier to impose sufficiently exact artificial boundary conditions at a remote boundary.

However, with the enlargement of the computational subdomain, the number of computations increases. Therefore, the problem of artificial boundary conditions consists in placing an artificial boundary as close as possible to the boundary of the neighborhood (of interest) of the body in a flow, for preserving the same accuracy.

In addition to flow problems, there are a number of other stationary problems of mathematical physics requiring the selection of a computational subdomain and the construction of artificial boundary conditions on its boundary. Among such problems we can point out, for example, the problem of diffraction of acoustic, electromagnetic, or elastic waves by a body, the problem of strong deformation of a solid in the neighborhood of a strong localized action (stamping) and many other classical problems.

For the construction of artificial boundary conditions on the basis of MDP, it is necessary to assume that differential and difference equations outside a bounded computational subdomain, as well as boundary conditions at infinity, are linear and homogeneous. In this case, artificial boundary conditions for the computational subdomain coincide, obviously, with the linear homogeneous boundary conditions that represent an equivalent replacement of the given conditions at infinity for the equations outside the computational subdomain.

Similar conditions were constructed for differential boundary-value problems in terms of differential potentials and boundary projections in Sect. 1.1 of Part II (Theorem 1.1.6). They can be written as $u_\Gamma - P_\Gamma^- u_\Gamma = 0$.

For difference boundary-value problems such conditions were constructed in terms of difference potentials and boundary projections in Sect. 2.1 of Part II (Theorem 2.1.6). These conditions can be written as $u_\gamma - P_\gamma^- u_\gamma = 0$.

We have introduced the minus sign in the notation of the boundary projections P_Γ^- and P_γ^- so as to stress that for the domains D and N in the corresponding theorems we take the domains D^- and N^-, which are the complements of the computational subdomains D^+ and N^+, respectively, so that, for example, $N^- = N^0 \setminus N^+$.

For computations, we apply artificial boundary conditions for difference problems approximating the corresponding differential problems. The degree of convenience of these artificial boundary conditions for computations depends on how convenient for computations is the difference auxiliary problem that is used to define the difference potential and the difference projection.

In this part of the book we restrict ourselves to the account of convenient artificial boundary conditions and their applications in the case of stationary problems of the flow of a homogeneous compressible and incompressible viscous or inviscid gas around the body.

For flow problems, we construct efficient artificial boundary conditions based on the following three heuristic considerations.

The first consideration consists in the fact that the system of equations of gas dynamics (1.1), (1.2) admits a sufficiently exact linearization outside the computational subdomain on a nonperturbed oncoming flow. The resulting linearized system turns out to be a system of equations with constant coefficients, since linearization was accomplished on a homogeneous flow. Bounded linearization is possible due to the fact that the perturbations of the homogeneous flow induced by the body become small at some distance from the body.

The second consideration is that the original flow problem can be replaced by one periodic in y with some period Y. The solution of a periodic problem in a fixed neighborhood of the body will be closer to the desired solution as the period Y increases, i.e., as the other bodies appearing in the periodic structure are removed.

The third consideration consists in the fact that the periodic problem under study can also be slightly changed by the insertion of a small parameter $\varepsilon > 0$. This parameter must be introduced so that, as $\varepsilon \to +0$, the solution $u(x, y, Y, \varepsilon)$ of the perturbed periodic problem tends to the solution of the unperturbed periodic problem in any fixed bounded subdomain. At the same time, the small parameter $\varepsilon > 0$ must be introduced so that the perturbed problem is convenient for the construction of artificial boundary conditions.

These three heuristic considerations were first put forward and used for the construction of artificial boundary conditions and the numerical solution of flow problems in [69].

In Chap. 1 we state and explain an algorithm for constructing efficient artificial boundary conditions using a comparatively simple example. This example can serve as a model for corresponding flow problems.

Chapter 2 written by S. V. Tsynkov is a survey of numerical results obtained for problems of gas dynamics using artificial boundary conditions constructed on the basis of MDP. It also contains comparisons of these results with those obtained on the basis of other approaches.