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Principles of Advanced Mathematical Physics

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Preface to Volume II

The first eleven chapters in this volume, 18 through 28, contain material that was developed in the third year of the three-year mathematical physics sequence at the University of Colorado. The central concepts are groups, manifolds, and differential geometry. I wish to thank Professors Wesley Brittin and Russel Dubisch for extensive discussions of this material, and I wish to thank Professor Wolf Beiglböck for advice and suggestions on the overall plan and on the material on group representations.

The material in the last three chapters, related broadly to recent work in differentiable dynamical systems, has been discussed in special courses on hydrodynamic stability and seminars on mathematical physics. That material is somewhat less well organized than the older subjects, but has been included because it contains various concepts of great potential value in physical science.

Boulder, August 1981

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