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For Helen and Wyeth

Foreword

The representation theory of locally compact groups has been vigorously developed in the past twenty-five years or so; of the various branches of this theory, one of the most attractive (and formidable) is the representation theory of semi-simple Lie groups which, to a great extent, is the creation of a single man: Harish-Chandra. The chief objective of the present volume and its immediate successor is to provide a reasonably self-contained introduction to Harish-Chandra's theory. Granting certain basic prerequisites (cf. infra), we have made an effort to give full details and complete proofs of the theorems on which the theory rests.

The structure of this volume and its successor is as follows. Each book is divided into chapters; each chapter is divided into sections; each section into numbers. We then use the decimal system of reference; for example, 1.3.2 refers to the second number in the third section of the first chapter. Theorems, Propositions, Lemmas, and Corollaries are listed consecutively throughout any given number. Numbers which are set in fine print may be omitted at a first reading. There are a variety of Examples scattered throughout the text; the reader, if he is so inclined, can view them as exercises ad libitum. The Appendices to the text collect certain ancillary results which will be used on and off in the systematic exposition; a reference of the form A2.4, say, would mean the fourth number in the second appendix. Numbers in square brackets after an individual's name designate works in the Bibliography found at the end of the volume; a list of notational conventions can be found there too (the reader is also advised to glance through Appendices 1 and 2 where additional commonly employed notations are introduced).

The basic prerequisites for reading this work (aside from elementary algebra and analysis) are as follows. The reader who is interested in the structure theory of semi-simple Lie groups and algebras, as presented in Chapters 1 and 2, will have to have background commensurate with Helgason [2], Hochschild [1] and Jacobson [3]. A little sheaf theory (cf. Hirzebruch [3]) is needed in Chapter 3 where the Borel-Weil Theorem is proved. Chapters 4, 5, 6, and 7 comprise an introduction to general group representation theory and spherical functions (with applications to semi-simple Lie groups); the reader who would like to put off the study of the first three chapters and instead move directly to representation theory could easily begin with Chapter 4 – the only prerequisites for this would be functional analysis in general along with distribution theory (cf. Schwartz [1], [2]) and elementary harmonic analysis (cf. Hewitt and Ross [1]) in particular (a familiarity with the contents of the book of Dixmier [15] would be helpful for Chapter 7). Chapters 8, 9, and 10 deal exclusively with analysis on semi-simple Lie groups and all the prerequisites cited above are needed for their study.

In preparing this work, I have drawn freely on the published and unpublished literature. Identification of sources is ordinarily not made in the text proper; however, in the Guide to the Literature at the end of the volume, the reader will find the works listed on which the exposition in any particular number is based. We apologize in advance for possible incompleteness and omissions in the list. In addition, the author makes no claim whatsoever to originality in the proofs or mode of presentation and assumes, of course, full responsibility for any errors or mistakes.

I have lectured on this material for a number of years at the University of Washington. I want to thank my students Pat Donaly, Helaman Ferguson, Harry Glover, Scott Osborne, and Cary Rader for helping me read the manuscript in its various stages of preparation. My colleagues here at the University, Professors Ramesh Gangolli, Vilnis Ozols, and Roger Richardson, have been most generous with their time and have made many improvements in the text. Elsewhere, Professors A. Borel, C. Chevalley, J.G.M. Fell, R. Goodman, Harish-Chandra, S. Helgason, R. Jewitt, R.P. Langlands, H. Leptin, J.G.M. Mars, S. Rallis, L. Robertson, P. Sally, G. Schiffmann, R. Steinberg, and N. Wallach have all been helpful; in particular the numerous and detailed conversations which I had with Professor Harish-Chandra in the academic year 1969–70 at the Institute for Advanced Study were decisive in the later stages of the preparation of this work. I am also happy to express here my gratitude to the authorities of the National Science Foundation for providing me with support over the past few years. Finally I want to thank my indomitable typists Susan Lowrey, Susan Seeds, and Helen Warner for the great job which they did.

Prologue

We shall give here a brief overview of the subject matter which forms the content of the present volume and its immediate successor. A more technical survey of the field can be found in Harish-Chandra [34] (see also Schiffmann's Séminaire Bourbaki article (number 323)).

Let G be a locally compact unimodular group satisfying the second axiom of countability and which, moreover, is postliminaire; let $\hat{G}$ be the set of unitary equivalence classes of irreducible unitary representations of G equipped with the hull-kernel topology; fix a Haar measure d_G on G – then, according to a well-known theorem of I.E. Segal, there exists a unique positive measure μ on $\hat{G}$ such that

$$\int_G |f(x)|^2 d_G(x) = \int_{\hat{G}} \operatorname{tr}(\hat{U}(f)\hat{U}(f)^*) d\mu(\hat{U})$$

for all $f \in L^1(G) \cap L^2(G)$. The measure μ is called the Plancherel measure for $\hat{G}$ (associated with the given Haar measure on G). This being so, the basic problem in 'L^2 Harmonic Analysis' on G is the 'explicit' determination of μ. [To illustrate, suppose that G is compact; normalize d_G by the requirement $\int_G d_G = 1$ – then the abstract theory tells us that μ assigns to each point $\hat{U}$ in $\hat{G}$ mass $\dim(\hat{U})$. In many concrete situations one can actually compute $\dim(\hat{U})$ in terms of the group structure on G and thereby obtain an 'explicit' description of μ; this will be the case, for instance, when G is a compact connected semi-simple Lie group (Weyl's dimension formula . . .).]

Let us suppose that G is a Lie group. For an arbitrary G, very little is known about $\hat{G}$ and no attempt has yet been made to attack the above problem; on the other hand, in the two extreme cases of a nilpotent G (or even a solvable G) and a semi-simple G, great progress has been made on this question. Indeed, the work of Dixmier, Kirillov, and Pukanszky gives the explicit Plancherel formula in the nilpotent case whereas the labors of Harish-Chandra give the Plancherel formula in the semi-simple case. In the present work we shall discuss in detail the methods which lead to Harish-Chandra's Plancherel Theorem for a semi-simple G.

To be more precise, let us now assume that G is a connected semi-simple Lie group with finite center; fix a maximal compact subgroup K of G – then the first results in the Harmonic Analysis on G serve to relate the representation theory of K (which is known . . .) with the representation theory of G. Thus let $\hat{K}$ be the set of unitary equivalence classes of irreducible unitary representations of K; fix a class δ in $\hat{K}$ – then it can be shown that the number of times that δ occurs in the restriction to K of an arbitrary irreducible unitary representation of G is $\leqslant \dim(\delta)$ (the dimension of δ). This fact has a number of important consequences. In the first place an easy argument then leads at once to the conclusion that G is postliminaire (hence is type I). A second consequence is this: Let U be an irreducible unitary representation of G – then, for every $f \in C_c^\infty(G)$, the operator $U(f)$ $(= \int_G f(x)U(x)d_G(x))$ is of the trace class and, moreover, the map T_U defined by the rule

$$f \mapsto \operatorname{tr}(U(f)) \qquad (f \in C_c^\infty(G))$$

is a distribution on G (in the sense of Laurent Schwartz). The distribution T_U is called the character of U; it is not difficult to prove that T_U determines U to within unitary equivalence. Furthermore, the general theory of central eigendistributions of $\mathfrak{Z}$ on G ($\mathfrak{Z}$ the algebra of differential operators on G which commute with both left and right translation), when applied to a given T_U, serves to imply that there exists a locally summable function F_U (say) which is analytic on the set of regular elements in G and with the property that

$$T_U(f) = \int_G f(x)F_U(x)d_G(x)$$

for all $f \in C_c^\infty(G)$; otherwise said, in the sense of distribution theory, the character of a given irreducible unitary representation of G is a function.

In order to compute the Plancherel measure μ, one must have at hand a large supply of irreducible unitary representations of G. [For the purpose of determining μ, it is not necessary, however, to find 'all' the irreducible unitary representations of G; for, due to the existence of 'complementary series', the support $\hat{G}_r$ of μ in $\hat{G}$, i.e. the reduced dual of G, will in general be a proper subset of $\hat{G}$.] To begin with, one has to isolate the discrete series for G (since irreducible square integrable unitary representations always have non-zero Plancherel measure). In his well-known Acta paper Harish-Chandra [30] completely solves this deep and formidable problem; it turns out that the discrete series for G is not empty iff $\operatorname{rank}(G) = \operatorname{rank}(K)$ and, when that is so, the (formal) dimension of a given irreducible square integrable unitary representation can be explicitly computed in terms of parameters involving a compact Cartan subgroup of G (the formula for the dimension (= formal degree) being strikingly analogous to that of Weyl's in the compact case). The other representations of G which enter into the Plancherel Theorem are constructed via the mechanism of 'induced representation'. Let P be a parabolic subgroup of G which is, moreover, cuspidal; let $P = MAN$ be a Langlands decomposition for P 'compatible' with K (thus the split component A of P is the 'vector' part of a Cartan subgroup of G, P being cuspidal); let σ be a representation in the discrete series for M, ν a unitary character of A – then the representation $U^{\sigma,\nu}$ unitarily induced from P to G by $\sigma \otimes \nu$ ($\sigma \otimes \nu(mhn) = h^\nu\sigma(m)$, $mhn \in MAN$) is irreducible 'in general' (those $U^{\sigma,\nu}$ which are not irreducible split into finitely many irreducible constituents and will not 'contribute' to μ). [One cannot, of course, carry out the above construction 'explicitly' unless the discrete series for M is known. Now M is, in general, neither connected nor semi-simple, but is always reductive; this is one of the reasons why it is necessary to develop the theory for reductive groups rather than just semi-simple groups.] To get the representations in the 'continuous series', one proceeds as follows. Let $J^1, \ldots, J^r$ be a maximal collection of mutually non-conjugate θ-stable Cartan subgroups of G (θ the Cartan involution per the pair (G,K)); let A_i be the vector part of J^i, $P_i = M_iA_iN_i$ a (cuspidal) parabolic subgroup of G with split component A_i – then, in an obvious notation, the irreducible unitary representations of G (apart from the discrete series, when it exists) which figure in the Plancherel Theorem are the irreducible U^{σ_i,ν_i}. Because the discrete series representations σ_i are in a close correspondence with the characters of the unitary dual of the compact part of J^i, it follows that $\hat{G}_r$ (the support of μ in $\hat{G}$) is in a close correspondence with the disjoint union $\hat{J}^1 \cup \ldots \cup \hat{J}^r$ in a sense which can be made precise.

The proof of the Plancherel Theorem itself is based on the 'philosophy of cusp forms': Eisenstein integral, c-function, wave packet, Mass-Selberg relations etc. These topics will be dealt with systematically in a later volume. [For an expository account of these developments consult Harish-Chandra [36].] In the present volume and its immediate successor we shall go far enough into the theory to get the discrete series for G (cf. Chapter 10). For an example, the reader may wish to turn to the Epilogue where the Plancherel Theorem is established via the classical method of 'integration by parts' in the special case when rank $(G/K) = 1$ (the general case is best approached, however, via the philosophy alluded to above).

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