An Introduction to Compactness Results in Symplectic Field Theory

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Preface

This text originates from special topics lectures I gave at Michigan State University on Symplectic Field Theory during the Fall of 2005 and the Spring of 2006 for graduate students after their third year. The first lecture covered compactness results, while the second was about polyfold theory.

Symplectic Field Theory yields powerful invariants for symplectic and contact manifolds. It is constructed using suitable moduli spaces of pseudoholomorphic curves, and it generalizes Floer Homology, Gromov–Witten theory and Contact Homology. The first paper on Symplectic Field Theory (SFT) was the 113-page survey by Yakov Eliashberg, Alexandre Givental and Helmut Hofer in the year 2000 [20]. As of now, a decade later, the general theory of SFT is still in development.

The concept of a polyfold was introduced by H. Hofer, K. Wysocki and E. Zehnder to address the numerous technical challenges in SFT in a systematic way. The reader is referred to the articles [39–42]. As a first application of polyfold theory H. Hofer, K. Wysocki and E. Zehnder recently gave a complete construction of Gromov–Witten theory in full generality [43]. Pseudoholomorphic curves are solutions to a nonlinear version of the Cauchy Riemann equations. Before a solution space of a nonlinear system of elliptic partial differential equations can be equipped with the structure of a polyfold it is necessary to understand its compactness properties. In the case of pseudoholomorphic curves this is the subject of this lecture.

Pseudoholomorphic curves have become a useful tool in symplectic geometry, and they were introduced by M. Gromov in his ground breaking 1985 paper [30]. Gromov's work is based on understanding moduli spaces of pseudoholomorphic curves on compact Riemann surfaces in a compact symplectic manifold. The subject of this text is to construct and describe the compactification relevant for SFT by F. Bourgeois, Y. Eliashberg, H. Hofer, K. Wysocki and E. Zehnder [12], a generalization of Gromov's result. Finally, the theory of polyfolds provides a general analytic framework for certain spaces which admit no smooth manifold structure, such as compactifications of spaces of pseudoholomorphic curves.

Andreas Floer was the first to recognize the importance of pseudoholomorphic curves on noncompact Riemann surfaces in his celebrated work on the Arnold conjecture [27] in 1988. Since then numerous different flavors of Floer Homology have been studied, and SFT is a construction in the same spirit.

Another important step consists of the work of Helmut Hofer in 1993 [33] on the Weinstein conjecture in dimension three. The Weinstein conjecture states that the Reeb vector field on any closed contact manifold has a periodic orbit. The main tool in H. Hofer's paper are pseudoholomorphic curves on the complex plane into the symplectization of a contact manifold M. In dimension three the Weinstein conjecture was proved by Clifford Taubes [67, 68] in 2007/2009 using Seiberg Witten equations. Interestingly, the gauge theoretic and the pseudoholomorphic curve stories are closely related. This is apparent from the proof that Seiberg–Witten Floer Homology and M. Hutching's Embedded Contact Homology (in some sense a version of SFT) are isomorphic (see [52, 69–73] for more information on ECH).

Hofer showed that the existence of a nonconstant pseudoholomorphic plane with finite energy implies the existence of a periodic orbit of the Reeb vector field, and he proved such existence results under some additional assumptions on M (see also [3] for other developments).

In the last decade special cases of the general Symplectic Field Theory construction and different flavors of it have been established and studied, already with far reaching applications. See [17, 18, 21–24, 44, 45, 52, 53] for a sample of the already large number of works on the subject.

In this text we will give a proof of the compactness results in SFT as in the paper [12], but with considerably more details and background material. We also present a version for curves with boundary (see [16, 26] for related results). The SFT compactness result describes what a sequence of pseudoholomorphic curves converges to (in a suitable sense), and it provides a description of the compactified moduli space. The outcome of this compactification procedure is the space of all holomorphic buildings which we discuss in detail. We also present all the necessary background material from hyperbolic geometry of surfaces. The purpose is to give a unified and detailed presentation which is currently not available in the literature. Hopefully this text makes the beginnings of Symplectic Field Theory more accessible.

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