

Editors

Timothy J. Barth
Michael Griebel
David E. Keyes
Risto M. Nieminen
Dirk Roose
Tamar Schlick

Michael Bader

Space-Filling Curves

An Introduction with Applications
in Scientific Computing

 Springer

Michael Bader
Department of Informatics
Technische Universität München
Germany

ISSN 1611-0994

ISBN 978-3-642-31045-4

ISBN 978-3-642-31046-1 (eBook)

DOI 10.1007/978-3-642-31046-1

Springer Heidelberg New York Dordrecht London

Library of Congress Control Number: 2012949128

Mathematics Subject Classification (2010): 68W01, 14H50, 65Y05, 65Y99, 68U99, 90C99

© Springer-Verlag Berlin Heidelberg 2013

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

Printed on acid-free paper

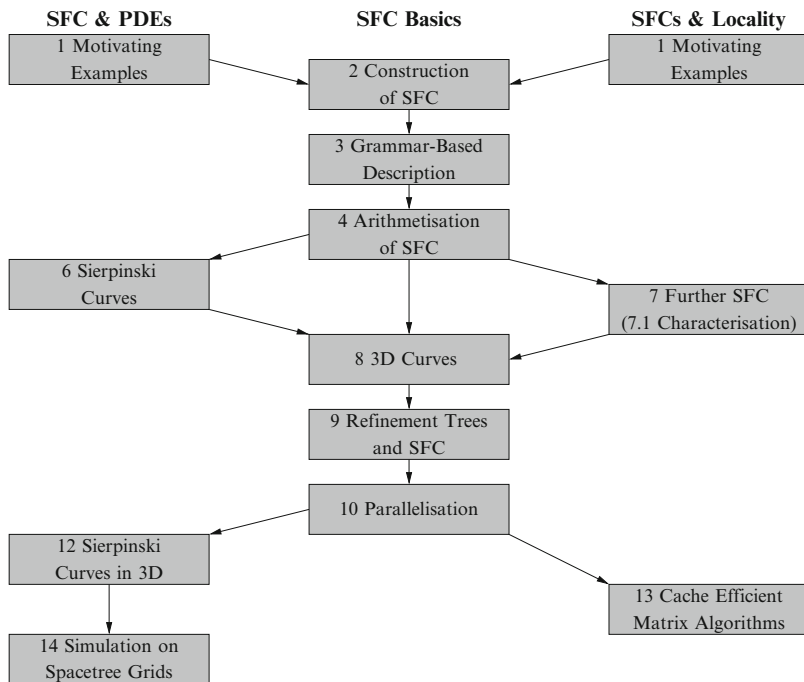
Springer is part of Springer Science+Business Media (www.springer.com)

Preface

Space-filling curves started their “lives” as mathematical curiosities, at the end of the nineteenth century. The idea that a one-dimensional curve may completely cover an area or a volume was, at that time, completely novel and counter-intuitive. Together with their relatives – such as Koch curves, the Cantor Set, and similar constructions – space-filling curves contradicted the then existing notion of a curve and of continuity and differentiability of functions (being continuous everywhere, but nowhere differentiable), and demanded new concepts in set theory and dimensionality. As a result, space-filling curves became almost notorious as “topological monsters”, and were studied by a good number of highly influential mathematicians, such as Peano, Hilbert, Lebesgue, Sierpinski, and others. The book of Hans Sagan [233] provides an excellent introduction and overview of these mathematical aspects of space-filling curves, as well as on their history.

The recursive and self-similar construction of space-filling curves leads to important locality properties – to put it in a nutshell, a space-filling-curve mapping will ensure that close-by parameters will be mapped to neighbouring points in the target set and – at least to a good extent – vice versa. It turned out that these properties are highly useful in the computational sciences. Algorithms based on space-filling curves can thus be used to construct spatial partitions to distribute a problem to different processors of a parallel computer, to improve the memory access behaviour of algorithms, or in general to find efficient data structures for multidimensional data. Most of the time, space-filling curves are not necessarily the best algorithm for the job where they are used – however, as Bartholdi and Platzman [32] put it, they are like “a good pocketknife”, being “simple and widely applicable”, and able to provide a convenient and satisfactory solution with comparably small effort in implementation and computation.

The aim of this book is therefore to give an introduction to the *algorithmics* of space-filling curves – to the various ways of describing them, and how these different description techniques lead to algorithms for different computational tasks. The focus will be on algorithms and applications in scientific computing, with a certain preference on mesh-based methods for partial differential equations.



How to Read This Book

The present book will hopefully serve multiple purposes – as a monograph on space-filling curves; as a reference to look up specific aspects, algorithms, or techniques to describe a specific curve; or as a textbook to be used as supplementary material for a course related to scientific computing, or even for a series of lectures that is dedicated to space-filling curves in particular. As a consequence, the sequential order of chapters I chose for this book will necessarily not be able to match the needs of every reader or lecturer. You are thus encouraged to read through this book in a non-sequential way, skip chapters or sections, do detours to later chapters, or similar. Some suggestions for selecting chapters and placing a different focus during a course (“just the basics” vs. techniques for partial differential equations vs. special focus on locality properties) are given in the figure above. In addition, every chapter will end with a box called “What’s next?” that will give some suggestions on where to read on.

Additional material (solution to exercises, code examples, links to other material, errata if necessary) will be published on the website

www.space-filling-curves.org.

Acknowledgements

The topic of space-filling curves has been a part of my research and teaching work for the last 8 years, at least, and numerous colleagues and fellow researchers provided valuable input and ideas. I would especially like to thank all my colleagues at Technische Universität München and at Universität Stuttgart – while I cannot list all of them, I want to express my special gratitude to those who have been my colleagues for the longest time: Hans Bungartz, Miriam Mehl, Tobias Neckel, Tobias Weinzierl, and Stefan Zimmer. Amongst all colleagues from other research groups, I would like to give thanks and tribute to Jörn Behrens, Herman Haverkort, Bill Mitchell, and Gerhard Zumbusch, who provided lots of inspiration for this book – especially for the chapters on Sierpinski curves and on locality properties.

Finally, and most of all, I want to thank Christoph Zenger, my former academic supervisor and mentor. In 2003, we initiated a joint lecture on *Algorithms in Scientific Computing*, with space-filling curves being one of three major topics. It was the first course, in which I was solely responsible for the content of a lecture, it has been running (with certain topical updates, of course) in this form for 8 years, and it was the starting-point and foundation for this book project.

Munich April 23, 2012

Michael Bader

Contents

1	Two Motivating Examples: Sequential Orders on Quadrees and Multidimensional Data Structures	1
1.1	Modelling Complicated Geometries with Quadrees, Octrees, and Spacetrees	1
1.1.1	Quadrees and Octrees	3
1.1.2	A Sequential Order on Quadtree Cells	3
1.1.3	A More Local Sequential Order on Quadtree Cells	6
1.2	Numerical Simulation: Solving a Simple Heat Equation	7
1.3	Sequentialisation of Multidimensional Data	9
1.3.1	Requirements for Efficient Sequential Orders	11
1.3.2	Row-Major and Column-Major Sequentialisation	12
2	How to Construct Space-Filling Curves	15
2.1	Towards a Bijective Mapping of the Unit Interval to the Unit Square	15
2.2	Continuous Mappings and (Space-Filling) Curves	17
2.3	The Hilbert Curve	18
2.3.1	Iterations of the Hilbert Curve	18
2.3.2	Approximating Polygons	19
2.3.3	Definition of the Hilbert Curve	20
2.3.4	Proof: h Defines a Space-Filling Curve	22
2.3.5	Continuity of the Hilbert Curve	23
2.3.6	Moore's Version of the Hilbert Curve	24
2.4	Peano Curve	25
2.5	Space-Filling Curves: Required Algorithms	27
3	Grammar-Based Description of Space-Filling Curves	31
3.1	Description of the Hilbert Curve Using Grammars	31
3.2	A Traversal Algorithm for 2D Data	34
3.3	Grammar-Based Description of the Peano Curve	37
3.4	A Grammar for Turtle Graphics	39

4	Arithmetic Representation of Space-Filling Curves	47
4.1	Arithmetic Representation of the Hilbert Mapping	47
4.2	Calculating the Values of h	49
4.3	Uniqueness of the Hilbert Mapping	52
4.4	Computation of the Inverse: Hilbert Indices	55
4.5	Arithmetisation of the Peano Curve	57
4.6	Efficient Computation of Space-Filling Mappings	59
4.6.1	Computing Hilbert Mappings via Recursion Unrolling	60
4.6.2	From Recursion Unrolling to State Diagrams	61
5	Approximating Polygons	67
5.1	Approximating Polygons of the Hilbert and Peano Curve	67
5.2	Measuring Curve Lengths with Approximating Polygons	69
5.3	Fractal Curves and Their Length	70
5.4	A Quick Excursion on Fractal Curves	72
6	Sierpinski Curves	77
6.1	The Sierpinski-Knopp Curve	77
6.1.1	Construction of the Sierpinski Curve	77
6.1.2	Grammar-Based Description of the Sierpinski Curve ...	79
6.1.3	Arithmetisation of the Sierpinski Curve	80
6.1.4	Computation of the Sierpinski Mapping	81
6.2	Generalised Sierpinski Curves	82
6.2.1	Bisecting Triangles Along Tagged Edges	83
6.2.2	Continuity and Locality of Generalised Sierpinski Curves	85
6.2.3	Filling Triangles with Curved Edges	87
7	Further Space-Filling Curves	93
7.1	Characterisation of Space-Filling Curves	93
7.2	Lebesgue Curve and Morton Order	95
7.3	The H -Index	99
7.4	The $\beta\Omega$ -Curve	101
7.5	The Gosper Flowsnake	104
8	Space-Filling Curves in 3D	109
8.1	3D Hilbert Curves	109
8.1.1	Possibilities to Construct a 3D Hilbert Curve	109
8.1.2	Arithmetisation of the 3D Hilbert Curve	113
8.1.3	A 3D Hilbert Grammar with Minimal Number of Non-Terminals	114
8.2	3D Peano Curves	116
8.2.1	A Dimension-Recursive Grammar to Construct a 2D Peano Curve	116
8.2.2	Extension of the Dimension-Recursive Grammar to Construct 3D Peano Curves	117

8.2.3	Peano Curves Based on 5×5 or 7×7 Refinement	119
8.2.4	Towards Peano’s Original Construction	122
8.3	A 3D Sierpinski Curve	123
9	Refinement Trees and Space-Filling Curves	129
9.1	Spacetrees and Refinement Trees	129
9.1.1	Number of Grid Cells for the Norm Cell Scheme and for a Quadtree	131
9.2	Using Space-Filling Curves to Sequentialise Spacetre Grids	132
9.2.1	Adaptively Refined Spacetrees	134
9.2.2	A Grammar for Adaptive Hilbert Orders	135
9.2.3	Refinement Information as Bitstreams	137
9.3	Sequentialisation of Adaptive Grids Using Space-Filling Curves	138
10	Parallelisation with Space-Filling Curves	143
10.1	Parallel Computation of the Heat Distribution on a Metal Plate	143
10.2	Partitioning with Space-Filling Curves	146
10.3	Partitioning and Load-Balancing Based on Refinement Trees and Space-Filling Curves	149
10.4	Subtree-Based Load Distribution	150
10.5	Partitioning on Sequentialised Refinement Trees	153
10.5.1	Modified Depth-First Traversals for Parallelisation	153
10.5.2	Refinement Trees for Parallel Grid Partitions	155
10.6	Data Exchange Between Partitions Defined via Space-Filling Curves	157
10.6.1	Refinement-Tree Partitions Using Ghost Cells	159
10.6.2	Non-Overlapping Refinement-Tree Partitions	160
11	Locality Properties of Space-Filling Curves	167
11.1	Hölder Continuity of Space-Filling Curves	167
11.1.1	Hölder Continuity of the 3D Hilbert Curve	168
11.1.2	Hölder Continuity and Parallelisation	168
11.1.3	Discrete Locality Measures for Iterations of Space-Filling Curves	171
11.2	Graph-Related Locality Measures	172
11.2.1	The Edge Cut and the Surface of Partition Boundaries	173
11.2.2	Connectedness of Partitions	174
12	Sierpinski Curves on Triangular and Tetrahedral Meshes	181
12.1	Triangular Meshes and Quasi-Sierpinski Curves	181
12.1.1	Triangular Meshes Using Red–Green Refinement	181
12.1.2	Two-Dimensional Quasi-Sierpinski Curves	182
12.1.3	Red–Green Closure for Quasi-Sierpinski Orders	184

12.2	Tetrahedral Grids and 3D Sierpinski Curves	184
12.2.1	Bisection-Based Tetrahedral Grids	184
12.2.2	Space-Filling Orders on Tetrahedral Meshes	188
13	Case Study: Cache Efficient Algorithms for Matrix Operations	195
13.1	Cache Efficient Algorithms and Locality Properties	195
13.2	Cache Oblivious Matrix-Vector Multiplication	199
13.3	Matrix Multiplication Using Peano Curves	201
13.3.1	Block-Recursive Peano Matrix Multiplication	204
13.3.2	Memory Access Patterns During the Peano Matrix Multiplication	205
13.3.3	Locality Properties and Cache Efficiency	206
13.3.4	Cache Misses on an Ideal Cache	208
13.3.5	Multiplying Matrices of Arbitrary Size	210
13.4	Sparse Matrices and Space-Filling Curves	211
14	Case Study: Numerical Simulation on Spacetree Grids Using Space-Filling Curves	215
14.1	Cache-Oblivious Algorithms for Element-Oriented Traversals ...	215
14.1.1	Element-Based Traversals on Spacetree Grids	217
14.1.2	Towards Stack-Based Traversals	219
14.2	Implementation of the Element-Oriented Traversals.....	221
14.2.1	Grammars for Stack Colouring	222
14.2.2	Input/Output Stacks Versus Colour Stacks	222
14.2.3	Algorithm for Sierpinski Traversal.....	225
14.2.4	Adaptivity: An Algorithm for Conforming Refinement	226
14.2.5	A Memory-Efficient Simulation Approach for Dynamically Adaptive Grids	227
14.3	Where It Works: And Where It Doesn't	228
14.3.1	Three-Dimensional Hilbert Traversals	229
14.3.2	A Look at Morton Order.....	229
15	Further Applications of Space-Filling Curves: References and Readings	235
A	Solutions to Selected Exercises	239
A.1	Two Motivating Examples	239
A.2	Space-Filling Curves	239
A.3	Grammar-Based Description of Space-Filling Curves	240
A.4	Arithmetic Representation of Space-Filling Curves	241
A.5	Approximating Polygons	243
A.6	Sierpinski Curves	246
A.7	Further Space-Filling Curves	246
A.8	Space-Filling Curves in 3D	247
A.9	Refinement Trees and Space-Filling Curves	248

- A.10 Parallelisation with Space-Filling Curves 249
- A.11 Locality Properties of Space-Filling Curves 251
- A.12 Sierpinski Curves on Triangular and Tetrahedral Meshes 251
- A.13 Cache Efficient Algorithms for Matrix Operations 253
- A.14 Numerical Simulation on Spacetree Grids Using
Space-Filling Curves 253

- References**..... 257

- Index**..... 271