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Invariant Theory and Algebraic Transformation Groups VIII

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Modular Invariant Theory

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For Diane Mary Brennan, Ian Alexander Brennan, Colin Cameron Brennan,
Graham Harold James Brennan and Maggie Orion Cameron.

For Charlene Lynn Janeway and Megan Melinda Jane Wehlau.

Preface

At the time we write this book there are several excellent references available which discuss various aspects of modular invariant theory from various points of view: Benson [6]; Derksen and Kemper [26]; Neusel [85]; Neusel and Smith [86]; and Smith [103]. In this book, we concentrate our attention on the modular invariant theory of finite groups. We have included various techniques for determining the structure of and generators for modular rings of invariants, while attempting to avoid too much overlap with the existing literature. An important goal has been to illustrate many topics with detailed examples. We have contrasted the differences between the modular and non-modular cases, and provided instances of our guiding philosophies and analogies. We have included a quick survey of the elements of algebraic geometry and commutative algebra as they apply to invariant theory. Readers who are familiar with these topics may safely skip this chapter.

We wish to thank our principal collaborators over the years with whom we have had so much pleasure exploring this fascinating subject: Ian Hughes, Gregor Kemper, R. James Shank, John Harris as well as our students and friends, Jianjun Chuai, Greg Smith, Mike Roth, Brandon Fodden, Emilie Dufresne, Asia Matthews and Chester Weatherby. In particular we thank John Harris, R. James Shank, Jianjun Chuai, Mike Roth, Emilie Dufresne, Asia Matthews, Chester Weatherby and Tristram Bogart for reading draft chapters and pointing out errors and suggesting improvements. We also thank Marie-José Bertin for clarifying the history of her own work to us.

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Fredericton, New Brunswick — *H. E. A. Eddy Campbell*
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$\mathbb{N}$		the natural numbers, including 0.
$\mathbb{Z}$		the integers.
$\mathbb{Q}$		the rational numbers.
$\mathbb{R}$		the real numbers.
$\mathbb{C}$		the complex numbers.
$\mathbb{F}$		a field of characteristic $p \neq 0$.
$\mathbb{F}^{\times}$		$\mathbb{F} \setminus \{0\}$.
$\mathbb{F}_p$		the prime field of order p.
$\mathbb{K}$		a field of arbitrary characteristic.
$\overline{\mathbb{K}}$		the algebraic closure of $\mathbb{K}$.
$\otimes$		$\otimes_{\mathbb{K}}$ or $\otimes_{\mathbb{F}}$.
V		a vector space over $\mathbb{F}$ or $\mathbb{K}$, most often of dimension n.
$\mathrm{GL}(V)$		the general linear group of V.
$\mathrm{GL}_n(\mathbb{K})$		the linear automorphisms of $\mathbb{K}^n$.
$\mathrm{SL}(V)$		the special linear group of V.
$\mathrm{SL}_n(\mathbb{K})$		the linear automorphisms of $\mathbb{K}^n$ of determinant 1.
G		a group, often a subgroup of $\mathrm{GL}(V)$.
σ, τ		group elements.
$\sigma(v), \sigma \cdot v$	2	the image $v \in V$ under the action of σ.
V^G	2	the subspace of V fixed (point-wise) by the action of G.
G_X	2	the stabilizer or isotropy subgroup of $X \subset V$.
G_v	2	the stabilizer or isotropy subgroup of $v \in V$.
V^*		the hom-dual $\hom_{\mathbb{K}}(V, \mathbb{K})$.
x^I	3	the monomial $x_1^{i_1} \cdots x_n^{i_n}$ for $I = (i_1, \ldots, i_n\}$.
$\mathbb{F}[V]$	3	the coordinate ring of V.
$\mathbb{F}[V]^G$	4	the invariant ring.
$\mathrm{Tr}^G(f)$	6	the trace of $f \in \mathbb{F}[V]$.
$\mathrm{Tr}^G_H(f)$	6	the trace of $f \in \mathbb{F}[V]$ relative to a subgroup $H \subset G$.
$\mathrm{N}(f)^G$	6	the norm of f.
$\mathrm{N}^G_H(f)$	7	the norm of f relative to a subgroup $H \subset G$.
Gf	7	the orbit of f under G, i.e., $Gf := \{\sigma(f) \mid \sigma \in G\}$.
$\beta(G, V)$	8	the Noether degree bound for the invariant ring.
$\lvert G \rvert$		the order of G.
V^{σ}	8	the elements of V fixed by σ.
$\mathbb{K}(V)^G$	10	the field of fractions of $\mathbb{K}[V]^G$.
$\mathrm{Quot}(\mathbb{K}[V]^G)$	10	the field of fractions of $\mathbb{K}[V]^G$.
mV	11	the vector space $V^{\oplus m}$.
$\mathcal{P}\mathrm{ol}$	11	polarization.
$\mathcal{R}$	11	restitution.
$\deg_y(f)$	18	the degree of f in the variable y.
Δ_{σ}	22	the twisted derivation $\sigma - 1$.
$\mathcal{V}(T)$	26	the zero-set of the polynomials in T.
$\mathcal{I}(X)$	26	the ideal of the variety or set X.

$\sqrt{I}$	26	the radical of the ideal I.
$\mathrm{Spec}(S)$	27	the space of all prime ideals of S.
$\mathrm{MaxSpec}(S)$	27	the space of all maximal prime ideals of S.
R		a graded commutative ring with 1.
R_f	29	the localization of R at f.
$\mathcal{H}(R,\lambda)$	34	the Hilbert series of R.
$V/\!\!/G$	42	the quotient variety.
$(V/\!\!/G)_{\mathrm{good}}$	56	the "good" locus.
$\mathrm{LM}(f)$	83	the lead monomial of f (with respect to some ordering).
$\mathrm{LT}(f)$	83	the lead term of f.
$\mathrm{LT}(R)$	85	the lead term algebra of R.
$\mathbb{K}[V]_G$	99	the ring of coinvariants.
$G_a(\mathbb{C})$	125	the additive group of $\mathbb{C}$.
$\mathrm{RL}_n(\mathbb{F})$	150	the subgroup of $\mathrm{GL}_n(\mathbb{F})$ generated by its reflections.
$\mathrm{GU}_n(\mathbb{F})$	150	the unitary group of dimension n over $\mathbb{F}$.
$\mathrm{RU}_n(\mathbb{F})$	150	the subgroup of $\mathrm{GU}_n(\mathbb{F})$ generated by its reflections.
$\mathrm{GO}_n(\mathbb{F})$	150	the general orthogonal group over $\mathbb{F}$.
$\mathrm{SO}_n(\mathbb{F})$	150	the special orthogonal group over $\mathbb{F}$.
$\Omega_n^{\pm}(\mathbb{F})$	150	the commutator subgroup of $\mathrm{GO}_n(\mathbb{F})$.