Part V

Stochastic Analysis in Infinite Dimensions

Stochastic analysis plays an eminent role in infinite dimension; the main applications of the theory in finite dimension remain valid; furthermore, in infinite dimension, the methods of classical analysis (Fourier transform, pseudo-differential operators) collapse so that the theory of elliptic PDE can be approached essentially only through stochastic analysis. We cannot give here a global overview of such a rich field. We will restrict ourselves to some subjects which are directly linked to the previous chapters of this book.

The Ornstein-Uhlenbeck operator was a key ingredient in Chapter II; we will study its associated process from which we shall deduce another approach to the regularity of laws studied in Chapter III; this approach to the regularity of laws has greater potential than those considered in Chapter III and Chapter VIII and for this reason will be developed in an axiomatic framework.

The infinitesimal transformations leaving quasi-invariant the probability measure can be considered as being the "tangent vector fields to the probability space". In the case of a Gaussian probability space, those vector fields have been considered in part I: all smooth vector fields taking their values in the Cameron-Martin space are tangent vector fields. As in finite dimensions, the theory of stochastic differential equations is a nonlinear theory, infinite dimensional stochastic analysis must be nonlinear. The broad generality carried over by this nonlinearity requirement is a precious laboratory for the elaboration of the concept of "tangent space to the probability space". In the case of the path space over a Lie group, this elaboration is possible under minor adjustments to the Wiener space case. On the contrary, the case of the path space over a Riemannian manifold needs the new concept of *tangent processes*. The lifting to the frame bundle produces an intrinsic measure-preserving map from the probability space of Brownian motion on $\mathbb{R}^l$ to the probability space of Brownian motion on a Riemannian manifold: this is the *Itô map*. Then the Itô map behaves as a diffeomorphism: it induces an isomorphism at the level of tangent processes.