Part III

Stochastic Integrals

The numerical model of Chapter I is associated to the Hilbert space ℓ^2 of square integrable sequences. Another remarkable Hilbert space is the space of square-integrable functions on a space with a measure of finite mass and without atoms. This Hilbert space has a sort of "continuous basis". The associated Gaussian probability space is called the *probability space of white noise*. The chaos decomposition over the white noise has a canonical expression, using what might be described as "Hermite polynomials with continuous indices", defined in terms of symmetric kernels. Those kernels are "integrated against the white noise", integrals which are divergent in absolute value, but become convergent by some kind of "continuous renormalization" of their Riemann-Lebesgue sums; those renormalized integrals are the *Wiener-Itô multiple stochastic integrals*.

The next issue is to compute stochastic integrals of random kernels; the possibility of proceeding to the required renormalization is guaranteed by the existence of first derivatives in the sense of Chapter I; the stochastic integral is equal to the divergence considered in then Chapter III. These two deep results establish a structural link between the theory of white noise stochastic integrals and the differential calculus considered in Part I.

An important case of white noise is the white noise associated to the Hilbert space of square integrable function on [0, 1] for the Lebesgue measure; this generates the *probability space of Brownian motion*. The multiplication operator by the indicator function $\mathbf{1}_{[0,t]}$ defines a one parameter family of Hilbert space projection operators Π_t; lifting this family Π_t to the probability space level we get the family of conditional expectations $E^{\mathcal{N}_t}$ relative to the *Itô filtration* $\mathcal{N}_t$. The family Π_t can be considered as a "homotopy operator" linking continuously the identity with the operator 0; at the level of the probability space, the homotopy is realized by the conditional expectations $E^{\mathcal{N}_*}$. It is well known in elementary differential calculus that a homotopy operator generates a Poincaré Lemma thereby making effective the computation of a differential form through its coboundary; on the probability space of the Brownian motion, the Poincaré Lemma is a formula with which to compute a functional from the knowledge of its gradient; iterating this formula, a stochastic Taylor formula is obtained. Classically, the study of Brownian martingales is done through the Itô calculus which becomes linked via this stochastic Taylor formula with the differential calculus of Part I.