Part I

Differential Calculus on Gaussian Probability Spaces

In the elementary theory of $\mathbb{R}^n$-valued random variables, operations on the subclass of random variables having a C^1-density relative to Lebesgue measure are often realized through computations of ordinary differential calculus: for instance, the determination of conditional laws by computing differential forms, the realization of a change of variables by computing Jacobians. Our purpose is to extend this methodology to more general probability spaces.

The Lebesgue measure of $\mathbb{R}^n$ can be characterized by its invariance under the group of translations. Given a probability space Ω, *the quasi-automorphism group* will be a "natural" group of transformations of Ω leaving quasi-invariant the probability measure; this notion is quite general; it will be developed in this book in the context of *a Gaussian probability space*, which means an abstract probability space Ω on which we have a Hilbert space $\mathcal{H}$ of Gaussian random variables. The additive group of $\mathcal{H}$ will define the quasi-automorphism group of Ω. Any unitary isomorphism of $\mathcal{H}$ will then generate an automorphism of the Gaussian probability space structure of Ω. The realization of this unitary invariance as a fact built into the construction of Ω itself is done in Chapter I.

The quasi-automorphism group $\mathcal{H}$ operates on a suitable algebra of random variables. The infinitesimal action of $\mathcal{H}$ will lead to the notion of *$\mathcal{H}$-Sobolev spaces* on Ω. Chapter II will be devoted to the study of the algebra of *smooth random variables* which are the random variables belonging to all those Sobolev spaces.

The *Jacobian* of an $\mathbb{R}^d$-valued smooth random variable is defined in Chapter III; an appropriate lower bound for this Jacobian will imply that the corresponding law has a C^∞-density relative to Lebesgue measure. This theorem will result from an interplay between classical harmonic analysis for Sobolev spaces on $\mathbb{R}^d$ and *elliptic estimates* established in Chapter II for Sobolev spaces on Ω. This interplay will be realized by lifting differential forms by the inverse image and pushing down by conditional expectations.