Chapter V. Multistep Methods for Stiff Problems

Multistep methods (BDF) were the first numerical methods to be proposed for stiff differential equations (Curtiss & Hirschfelder 1952) and since Gear's book (1971) computer codes based on these methods have been the most prominent and most widely used for all stiff computations.

This chapter introduces the linear stability theory for multistep methods in Sect. V.1, and arrives at the famous theorem of Dahlquist which says that A-stable multistep methods cannot have high order. Attempts to circumvent this barrier proceed mainly in two directions: either study methods with slightly weaker stability requirements (Sect. V.2) or introduce new classes of methods (Sect. V.3). Order star theory on Riemann surfaces (Sect. V.4) then helps to extend Dahlquist's barrier to generalized methods and to explain various properties of stability domains. Section V.5 presents numerical experiments with several codes based on the methods introduced.

Since all the foregoing stability theory is based uniquely on linear autonomous problems $y' = Ay$, the question arises of their validity for general nonlinear problems. This leads to the concepts of G-stability for multistep methods (Sect. V.6) and algebraic stability for general linear methods (Sect. V.9).

Another important subject is convergence estimates for $h \to 0$ which are independent of the stiffness (the analogue of B-convergence in Sect. IV.15). We describe various techniques for obtaining such estimates in Sections V.7 (for linear problems) as well as V.6 and V.8 (for nonlinear problems). These techniques are: use of G-stability, the Kreiss matrix theorem, the multiplier technique and, last but not least, a discrete variation of constants formula.