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Solving Ordinary Differential Equations II

Stiff and Differential-Algebraic Problems

Second Revised Edition
With 137 Figures



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To Evi and Myriam

From the Preface to the First Edition

“Whatever regrets may be, we have done our best.”
(Sir Ernest Shackleton, turning back on 9 January 1909 at 88° 23' South.)

Brahms struggled for 20 years to write his first symphony. Compared to this, the 10 years we have been working on these two volumes may even appear short.

This second volume treats stiff differential equations and differential algebraic equations. It contains three chapters: Chapter IV on one-step (Runge-Kutta) methods for stiff problems, Chapter V on multistep methods for stiff problems, and Chapter VI on singular perturbation and differential-algebraic equations.

Each chapter is divided into sections. Usually the first sections of a chapter are of an introductory nature, explain numerical phenomena and exhibit numerical results. Investigations of a more theoretical nature are presented in the later sections of each chapter.

As in Volume I, the formulas, theorems, tables and figures are numbered consecutively in each section and indicate, in addition, the section number. In cross references to other chapters the (latin) chapter number is put first. References to the bibliography are again by “author” plus “year” in parentheses. The bibliography again contains only those papers which are discussed in the text and is in no way meant to be complete.

It is a pleasure to thank J. Butcher, G. Dahlquist, and S.P. Nørsett (coauthor of Volume I) for their interest in the subject and for the numerous discussions we had with them which greatly inspired our work. Special thanks go to the participants of our seminar in Geneva, in particular Ch. Lubich, A. Ostermann and M. Roche, where all the subjects of this book have been presented and discussed over the years. Much help in preparing the manuscript was given by J. Steinig, Ch. Lubich and A. Ostermann who read and re-read the whole text and made innumerable corrections and suggestions for improvement. We express our sincere gratitude to them. Many people have seen particular sections and made invaluable suggestions and remarks: M. Crouzeix, P. Deuffhard, K. Gustafsson, G. Hall, W. Hundsdorfer, L. Jay, R. Jeltsch, J.P. Kauthen, H. Kraaijevanger, R. März, and O. Nevanlinna. . . . Several pictures were produced by our children Klaudia Wanner and Martin Hairer, the one by drawing the other by hacking.

The marvellous, perfect and never failing TEX program of D. Knuth allowed us to deliver a camera-ready manuscript to Springer Verlag, so that the book could be produced rapidly and at a reasonable price. We acknowledge with pleasure the numerous remarks of the planning and production group of Springer Verlag concerning fonts, style and other questions of elegance.

Preface to the Second Edition

The preparation of the second edition allowed us to improve the first edition by rewriting many sections and by eliminating errors and misprints which have been discovered. In particular we have included new material on

- methods with extended stability (Chebyshev methods) (Sect. IV.2);
- improved computer codes and new numerical tests for one- and multistep methods (Sects. IV.10 and V.5);
- new results on properties of error growth functions (Sects. IV.11 and IV.12);
- quasilinear differential equations with state-dependent mass matrix (Sect. VI.6).

We have completely reorganized the chapter on differential-algebraic equations by including three new sections on

- index reduction methods (Sect. VII.2);
- half-explicit methods for index-2 systems (Sect. VII.6);
- symplectic methods for constrained Hamiltonian systems and backward error analysis on manifolds (Sect. VII.8).

Our sincere thanks go to many persons who have helped us with our work:

- all readers who kindly drew our attention to several errors and misprints in the first edition, in particular C. Bendtsen, R. Chan, P. Chartier, T. Eirola, L. Jay, P. Kaps, J.-P. Kauthen, P. Leone, S. Maset, B. Owren, and L.F. Shampine;
- those who read preliminary versions of the new parts of this edition for their invaluable suggestions: M. Arnold, J. Cash, D.J. Higham, P. Kunkel, Chr. Lubich, A. Medovikov, A. Murua, A. Ostermann, and J. Verwer.
- the staff of the Geneva computing center and of the mathematics library for their constant help;
- the planning and production group of Springer-Verlag for numerous suggestions on presentation and style.

All figures have been recomputed and printed, together with the text, in Postscript. All computations and text processings were done on the SUN workstations of the Mathematics Department of the University of Geneva.

April 1996

The Authors

Contents

Chapter IV. Stiff Problems – One-Step Methods

IV.1	Examples of Stiff Equations	2
	Chemical Reaction Systems	3
	Electrical Circuits	4
	Diffusion	6
	A “Stiff” Beam	8
	High Oscillations	11
	Exercises	11
IV.2	Stability Analysis for Explicit RK Methods	15
	Stability Analysis for Euler’s Method	15
	Explicit Runge-Kutta Methods	16
	Extrapolation Methods	18
	Analysis of the Examples of IV.1	18
	Automatic Stiffness Detection	21
	Step-Control Stability	24
	A PI Step Size Control	28
	Stabilized Explicit Runge-Kutta Methods	31
	Exercises	37
IV.3	Stability Function of Implicit RK-Methods	40
	The Stability Function	40
	<i>A</i> -Stability	42
	<i>L</i> -Stability and <i>A</i> (α)-Stability	44
	Numerical Results	46
	Stability Functions of Order $\geq s$	47
	Padé Approximations to the Exponential Function	48
	Exercises	49
IV.4	Order Stars	51
	Introduction	51
	Order and Stability for Rational Approximations	56
	Stability of Padé Approximations	58
	Comparing Stability Domains	58
	Rational Approximations with Real Poles	61
	The Real-Pole Sandwich	62
	Multiple Real-Pole Approximations	67
	Exercises	70
IV.5	Construction of Implicit Runge-Kutta Methods	71
	Gauss Methods	71
	Radau IA and Radau IIA Methods	72

Lobatto IIIA, IIIB and IIIC Methods	75
The W -Transformation	77
Construction of Implicit Runge-Kutta Methods	83
Stability Function	84
Positive Functions	86
Exercises	89
IV.6 Diagonally Implicit RK Methods	91
Order Conditions	91
Stiffly Accurate SDIRK Methods	92
The Stability Function	96
Multiple Real-Pole Approximations with $R(\infty)=0$	98
Choice of Method	99
Exercises	100
IV.7 Rosenbrock-Type Methods	102
Derivation of the Method	102
Order Conditions	104
The Stability Function	108
Construction of Methods of Order 4	108
Higher Order Methods	111
Implementation of Rosenbrock-Type Methods	111
The "Hump"	113
Methods with Inexact Jacobian (W -Methods)	114
Exercises	117
IV.8 Implementation of Implicit Runge-Kutta Methods	118
Reformulation of the Nonlinear System	118
Simplified Newton Iterations	119
The Linear System	121
Step Size Selection	123
Implicit Differential Equations	127
An SDIRK-Code	128
SIRK-Methods	128
Exercises	130
IV.9 Extrapolation Methods	131
Extrapolation of Symmetric Methods	131
Smoothing	133
The Linearly Implicit Mid-Point Rule	134
Implicit and Linearly Implicit Euler Method	138
Implementation	139
Exercises	142
IV.10 Numerical Experiments	143
The Codes Used	143
Twelve Test Problems	144
Results and Discussion	152
Partitioning and Projection Methods	160
Exercises	165
IV.11 Contractivity for Linear Problems	167
Euclidean Norms (Theorem of von Neumann)	168
Error Growth Function for Linear Problems	169
Small Nonlinear Perturbations	172
Contractivity in $\ \cdot\ _\infty$ and $\ \cdot\ _1$	175
Study of the Threshold Factor	176

Absolutely Monotonic Functions	178
Exercises	179
IV.12 B-Stability and Contractivity	180
One-Sided Lipschitz Condition	180
<i>B</i> -Stability and Algebraic Stability	181
Some Algebraically Stable IRK Methods	183
<i>AN</i> -Stability	184
Reducible Runge-Kutta Methods	187
The Equivalence Theorem for <i>S</i> -Irreducible Methods	188
Error Growth Function	193
Computation of $\varphi_B(x)$	195
Exercises	199
IV.13 Positive Quadrature Formulas and B-Stable RK-Methods ..	201
Quadrature Formulas and Related Continued Fractions	201
Number of Positive Weights	203
Characterization of Positive Quadrature Formulas	205
Necessary Conditions for Algebraic Stability	206
Characterization of Algebraically Stable Methods	209
The “Equivalence” of <i>A</i> - and <i>B</i> -Stability	211
Exercises	213
IV.14 Existence and Uniqueness of IRK Solutions	215
Existence	215
A Counterexample	217
Influence of Perturbations and Uniqueness	218
Computation of $\alpha_0(A^{-1})$	220
Methods with Singular <i>A</i>	222
Lobatto IIIC Methods	223
Exercises	223
IV.15 B-Convergence	225
The Order Reduction Phenomenon	225
The Local Error	228
Error Propagation	229
<i>B</i> -Convergence for Variable Step Sizes	230
<i>B</i> -Convergence Implies Algebraic Stability	232
The Trapezoidal Rule	234
Order Reduction for Rosenbrock Methods	236
Exercises	237

Chapter V. Multistep Methods for Stiff Problems

V.1 Stability of Multistep Methods	240
The Stability Region	240
Adams Methods	242
Predictor-Corrector Schemes	244
Nyström Methods	245
BDF	246
The Second Dahlquist Barrier	247
Exercises	249
V.2 “Nearly” A-Stable Multistep Methods	250
<i>A</i> (α)-Stability and Stiff Stability	250
High Order <i>A</i> (α)-Stable Methods	251
Approximating Low Order Methods with High Order Ones	253

A Disc Theorem	254
Accuracy Barriers for Linear Multistep Methods	254
Exercises	259
V.3 Generalized Multistep Methods	261
Second Derivative Multistep Methods of Enright	261
Second Derivative BDF Methods	265
Blended Multistep Methods	266
Extended Multistep Methods of Cash	267
Multistep Collocation Methods	270
Methods of “Radau” Type	273
Exercises	275
V.4 Order Stars on Riemann Surfaces	279
Riemann Surfaces	279
Poles Representing Numerical Work	283
Order and Order Stars	284
The “Daniel and Moore Conjecture”	286
Methods with Property C	288
General Linear Methods	290
Dual Order Stars	295
Exercises	297
V.5 Experiments with Multistep Codes	300
The Codes Used	300
Exercises	304
V.6 One-Leg Methods and G-Stability	305
One-Leg (Multistep) Methods	305
Existence and Uniqueness	306
G -Stability	307
An Algebraic Criterion	309
The Equivalence of A -Stability and G -Stability	310
A Criterion for Positive Functions	313
Error Bounds for One-Leg Methods	314
Convergence of A -Stable Multistep Methods	317
Exercises	319
V.7 Convergence for Linear Problems	321
Difference Equations for the Global Error	321
The Kreiss Matrix Theorem	323
Some Applications of the Kreiss Matrix Theorem	326
Global Error for Prothero and Robinson Problem	328
Convergence for Linear Systems with Constant Coefficients	329
Matrix Valued Theorem of von Neumann	330
Discrete Variation of Constants Formula	332
Exercises	337
V.8 Convergence for Nonlinear Problems	339
Problems Satisfying a One-Sided Lipschitz Condition	339
Multiplier Technique	342
Multipliers and Nonlinearities	346
Discrete Variation of Constants and Perturbations	348
Convergence for Nonlinear Parabolic Problems	349
Exercises	354
V.9 Algebraic Stability of General Linear Methods	356
G -Stability	356

Algebraic Stability	357
AN -Stability and Equivalence Results	359
Multistep Runge-Kutta Methods	362
Simplifying Assumptions	363
Quadrature Formulas	365
Algebraically Stable Methods of Order $2s$	366
B -Convergence	368
Exercises	370

Chapter VI. Singular Perturbation Problems and Index 1 Problems

VI.1 Solving Index 1 Problems	372
Asymptotic Solution of van der Pol's Equation	372
The ε -Embedding Method for Problems of Index 1	374
State Space Form Method	375
A Transistor Amplifier	376
Problems of the Form $Mu' = \varphi(u)$	378
Convergence of Runge-Kutta Methods	380
Exercises	381
VI.2 Multistep Methods	382
Methods for Index 1 Problems	382
Convergence for Singular Perturbation Problems	383
Exercises	387
VI.3 Epsilon Expansions for Exact and RK Solutions	388
Expansion of the Smooth Solution	388
Expansions with Boundary Layer Terms	389
Estimation of the Remainder	391
Expansion of the Runge-Kutta Solution	392
Convergence of RK-Methods for Differential-Algebraic Systems	394
Existence and Uniqueness of the Runge-Kutta Solution	397
Influence of Perturbations	398
Estimation of the Remainder in the Numerical Solution	399
Numerical Confirmation	403
Perturbed Initial Values	405
Exercises	406
VI.4 Rosenbrock Methods	407
Definition of the Method	407
Derivatives of the Exact Solution	408
Trees and Elementary Differentials	409
Taylor Expansion of the Exact Solution	411
Taylor Expansion of the Numerical Solution	412
Order Conditions	415
Convergence	416
Stiffly Accurate Rosenbrock Methods	418
Construction of RODAS, a Stiffly Accurate Embedded Method	420
Inconsistent Initial Values	422
Exercises	424
VI.5 Extrapolation Methods	426
Linearly Implicit Euler Discretization	426
Perturbed Asymptotic Expansion	428
Order Tableau	431

Error Expansion for Singular Perturbation Problems	433
Dense Output	438
Exercises	441
VI.6 Quasilinear Problems	442
Example: Moving Finite Elements	442
Problems of Index One	445
Numerical Treatment of $C(y)y' = f(y)$	446
Extrapolation Methods	447
Exercises	448

Chapter VII. Differential-Algebraic Equations of Higher Index

VII.1 The Index and Various Examples	452
Linear Equations with Constant Coefficients	452
Differentiation Index	454
Differential Equations on Manifolds	457
The Perturbation Index	459
Control Problems	461
Mechanical Systems	463
Exercises	465
VII.2 Index Reduction Methods	468
Index Reduction by Differentiation	468
Stabilization by Projection	470
Differential Equations with Invariants	472
Methods Based on Local State Space Forms	474
Overdetermined Differential-Algebraic Equations	477
Unstructured Higher Index Problems	478
Exercises	480
VII.3 Multistep Methods for Index 2 DAE	481
Existence and Uniqueness of Numerical Solution	482
Influence of Perturbations	484
The Local Error	485
Convergence for BDF	486
General Multistep Methods	489
Solution of the Nonlinear System by Simplified Newton	490
Exercises	491
VII.4 Runge-Kutta Methods for Index 2 DAE	492
The Nonlinear System	492
Estimation of the Local Error	494
Convergence for the y -Component	496
Convergence for the z -Component	497
Collocation Methods	498
Superconvergence of Collocation Methods	500
Projected Runge-Kutta Methods	502
Summary of Convergence Results	504
Exercises	505
VII.5 Order Conditions for Index 2 DAE	506
Derivatives of the Exact Solution	506
Trees and Elementary Differentials	507
Taylor Expansion of the Exact Solution	508

Derivatives of the Numerical Solution	510
Order Conditions	512
Simplifying Assumptions	514
Projected Runge-Kutta Methods	515
Exercises	518
VII.6 Half-Explicit Methods for Index 2 Systems	519
Half-Explicit Runge-Kutta Methods	520
Extrapolation Methods	525
β -Blocked Multistep Methods	527
Exercises	529
VII.7 Computation of Multibody Mechanisms	530
Description of the Model	530
Fortran Subroutines	533
Computation of Consistent Initial Values	535
Numerical Computations	536
A Stiff Mechanical System	541
Exercises	542
VII.8 Symplectic Methods for Constrained Hamiltonian Systems ..	543
Properties of the Exact Flow	544
First Order Symplectic Method	545
SHAKE and RATTLE	548
The Lobatto IIIA-III B Pair	550
Composition Methods	554
Backward Error Analysis (for ODEs)	555
Backward Error Analysis on Manifolds	559
Exercises	562
Appendix. Fortran Codes	565
Driver for the Code RADAU5	566
Subroutine RADAU5	568
Subroutine RADAUP	574
Subroutine RODAS	574
Subroutine SEULEX	575
Problems with Special Structure	575
Use of SOLOUT and of Dense Output	576
Bibliography	577
Symbol Index	605
Subject Index	607