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Ralf Greve · Heinz Blatter

# Dynamics of Ice Sheets and Glaciers

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To Professor Kolumban Hutter,  
Professor Atsumu Ohmura  
and  
Professor Takeo Hondoh



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## Preface

Ice sheets, ice shelves, ice caps and glaciers are active, dynamic components of the climate system of the Earth, and they deserve the same scientific attention as the atmosphere and the oceans. However, while the dynamics of the atmosphere and the oceans have been studied intensively and literature on these topics abound, awareness of the importance of ice dynamics within the big picture has increased only recently. Just as an example, the widely acclaimed and valued book *Geophysical Fluid Dynamics* by Pedlosky (1987) states that “the subject has tended to focus on the dynamics of large-scale phenomena in the atmosphere and the oceans”, and, consequently, only these are presented in the book. On the other hand, *glaciology* is an established field of research, and glacier dynamics has been dealt with in the literature to some extent; however, with a certain focus on smaller-scale phenomena. Treatments of the large-scale dynamics of ice sheets are mainly found in the specialist literature.

In this book, we try to bridge the gap between the conventional understandings of geophysical fluid dynamics and glacier dynamics. Chapter 1 puts the subject into the wider context of climate research. In Chapter 2, the mathematical properties of vectors and tensors are reviewed briefly. Chapter 3 presents a solid, continuum-mechanical background, which is the foundation for the subject matter of the remainder of the book. This chapter goes further than the immediate needs of ice sheet and glacier dynamics in order to provide a framework applicable to a great variety of related problems in geophysics (including the above-mentioned large-scale dynamics of the atmosphere and the ocean), physics and engineering sciences. The material properties of polycrystalline ice, as it occurs in land ice masses on Earth, are discussed in Chapter 4. The core of this book is made up by Chapters 5–7, devoted to the dynamics of ice sheets, ice shelves and glaciers, respectively. Special emphasis is put on systematically developing hierarchies of approximations for the different systems, and suitable numerical solution techniques are discussed. Chapter 8 is concerned with simple models for glacial isostasy, the reaction of the solid Earth to temporally varying ice loads. In Chapter 9 some more advanced

and demanding topics of current research related to ice dynamics (induced anisotropy, compressible firn, polythermal glaciers) are treated. Chapter 10 concludes the book.

The content is based, largely, on lectures about ice-sheet/glacier dynamics and numerical models in glaciology developed by the authors over the past years. These lectures are offered jointly at the Hokkaido University, Sapporo, and the Swiss Federal Institute of Technology (ETH) Zurich, in association with the International Antarctic Institute (IAI), an international, multi-campus programme in cryosphere science education (<http://www.earth.ees.hokudai.ac.jp/IAI/>, <http://www.iai.utas.edu.au/>). The level of treatment caters mainly to graduate students, post-graduate students and researchers, but most of the material should also be understandable for motivated upper-level undergraduate students.

In order to eliminate one source of distraction especially for student readers, we have refrained from giving detailed references to original literature in the style of scientific articles. Instead, references have been kept at a reasonable minimum, and whenever possible, overview articles and textbooks have been given preference. An exception is Chapter 9, in which less well established topics are discussed, and which is more biased by the authors' own perspective than the material in the preceding chapters. The literature list at the end of the book is also understood as a suggestion for supplemental and/or further-reaching reading. We apologize to those colleagues whose publications are not quoted.

Hutter and Jöhnk (2004) end the preface of their book *Continuum Methods of Physical Modeling* with the statement that “writing a book can never be finished, a book has to be abandoned!” We cannot put it in better words. Abandoning this book is what we are now going to do, well knowing that it is not perfect, but nevertheless hoping that a variety of readers with backgrounds in glaciology, climate science, geophysical fluid dynamics, continuum mechanics, physics and applied mathematics will find it useful and inspiring in the years to come.

Sapporo, Japan; Zurich, Switzerland  
June 2009

*Ralf Greve*  
*Heinz Blatter*



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