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Introduction to Conformal Field Theory

With Applications to String Theory

 Springer

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Preface

These lecture notes are based on a graduate course given at the “Ludwig-Maximilians-Universität München” during the winter term 2007/2008 as part of the “Theoretical and Mathematical Physics” master programme[‡]. Although the main target group of this course were master students, we decided to prepare these notes for a more general audience including Ph.D. students and Postdocs.

These lecture notes are intended to give an introduction to conformal field theories in two dimensions with special emphasis on computational issues important for applications in string theory. We assume the reader to be familiar with Quantum Mechanics on the level of a graduate course and to have some basic knowledge of quantum field theory, even though the later is not a necessity. The notions of conformal field theory will be introduced in due course, however, string theory is not introduced in a self-contained manner. While familiarity with string theory is not a prerequisite for understanding these notes, for students intending to appreciate the presented techniques we highly recommend to study an introductory book on string theory in parallel.

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