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**Advanced Finite Element  
Method in Structural Engineering**

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# Advanced Finite Element Method in Structural Engineering

With 219 figures



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Yu-Qiu Long  
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# 新型有限元论

——结构工程中的高等有限元方法

## Advanced Finite Element Method in Structural Engineering



## 内 容 简 介

本书是在中文专著《新型有限元论》(2004年版)的基础上补充了2004年至2008年期间的新成果所撰写的英文专著,是龙驭球院士、岑松博士和龙志飞教授及其研究组多年来在新型有限元方面研究成果的系统论述。全书分为20章。除首尾两章外,其余18章分为3篇:第1篇是变分原理进展,介绍分区和含参变分原理2项成果;它们为构造新型有限元起到理论指导作用。第2篇是有限元法进展初论,重点介绍广义协调元;这是在协调元与非协调元之间另辟的新路,使协调问题和收敛问题得到合理解决,单元构造方案可以灵活优选,学科内容得到充实更新;广义协调元是新型有限元方面的主要成果,在本书中起核心作用。第3篇是有限元法进展续论,补充介绍4项成果,包括分区混合元法、解析试函数法、第一和第二类四边形面积坐标法和样条函数有限元法,在本书中起锦上添花作用。本书还结合7项成果的论述,介绍了总共108个相关的新单元。

本书可作为高等学校力学、土木、机械等专业研究生和高年级本科生的教材和参考书,也可供相关领域教师和科技人员参考。

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**Professor Yu-Qiu Long** (1926— ) is a Full Professor in the Department of Civil Engineering, Tsinghua University, and also a Member of the Chinese Academy of Engineering since 1995. During the past 60 years, he has been recognized as an expert in the areas of structural mechanics, shell structures, finite element method and variational principle. He was the first President of the Chinese Association of Structural Engineering between 1998 and 2003; Council Member of the Chinese Association of Computational Mechanics from 1957 to 1990; and Editor-in-Chief of a Chinese journal, *Engineering Mechanics*, between 1991 and 1999. He is currently a member of Editorial Boards of many international and Chinese journals, such as *Advances in Structural Engineering* (since 1997), *International Journal of Structural Stability and Dynamics* (since 2001), *Applied Mathematics and Mechanics* (since 1979), and so on. He published his first book on the finite element method in 1978, which is one of the earliest books on this subject in China and has a broad influence on Chinese readers.



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**Professor Zhi-Fei Long** (1957— ) is a professor in the School of Mechanics & Civil Engineering, China University of Mining & Technology (Beijing). He is an expert in finite element method and structural mechanics, and has published 4 books and more than 90 papers in related fields. He won the Science Award of China Ministry of Education (First Class, 1993), for the studies on the generalized energy principles and new finite element models. The textbook titled by *New Monograph of Finite Element Method* (written by Zhi-Fei Long and Song Cen, published by China Hydraulic and Water-power Press in 2001) has produced a broad influence in Chinese universities.

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# Preface

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The main purpose of this book is to describe some developments in finite element method and related variational principles. Since this book only deals with the areas the authors are familiar with, it is impossible to cover every aspect of these subjects. This book is composed of 20 chapters. Except for *Introduction* (Chap. 1) and *Concluding Remarks* (Chap. 20), in the other 18 chapters, seven theoretical achievements (two achievements in variational principles and five achievements in finite element methods) are introduced, which are subdivided into three Parts.

Part I focuses on advances in the variational principles. Two innovations in this subject are discussed here.

(1) *Sub-region variational principles* (Chap. 2). The concept of *sub-region* is introduced for establishing new variational principles suitable for the developments of the finite element method.

(2) *Variational principles with several adjustable parameters* (Chap. 3). Several adjustable parameters are included in the variational principles so that a broader optimization space is available.

Part II focuses on the main advances in the finite element method—generalized conforming elements (the third innovation). Eight chapters are employed to illustrate this innovation.

(3) *Generalized conforming elements* (Chaps. 4–11). Firstly, from the viewpoint of theory, the generalized conforming element opens a new way between conforming and non-conforming elements, so that the puzzle of the convergence problem for non-conforming elements can be rationally solved. Meanwhile, various new conforming schemes, including point conforming, line conforming, perimeter conforming, SemiLoof conforming, least square conforming and their combination forms, have been successfully proposed. Secondly, from the viewpoint of applications, the successful application of the generalized conforming element method was first realized for thin plate bending problem, in which a series of high performance thin plate element models were presented. Subsequently, the novel technique was successfully generalized to other fields, and a large number of new models, including membrane elements, membrane elements with drilling DOFs, thin-thick plate elements, laminated composite plate elements, flat-shell

elements, curved shell elements, etc., were also successfully constructed.

Part III focuses on the other advances in the finite element method. Eight chapters are employed to discuss four additional subject innovations.

(4) *Sub-region mixed element method* (Chaps. 12–13). It provides a novel solution strategy for fracture problem by complementarity and coupling of displacement-based element and stress-based element.

(5) *Analytical trial function method* (Chaps. 14–15). This method exhibits rewarding cooperation between analytical and discrete methods, and provides effective solution strategy for shear locking, trapezoidal locking, and singular stress problems.

(6) *Quadrilateral area coordinate method* (Chaps. 16–17). This method indicates that the area coordinate method is generalized from the traditional triangular element field to new fields.

(7) *Spline element method* (Chaps. 18–19). This method indicates that the advantages of the spline functions have been adopted by the finite element method.

While introducing above seven theoretical innovations, five new element series with 108 new element models, which were directly derived from the five achievements in FEM, are also discussed in detail or briefly (see Table 20.2). Furthermore, based on these developments, effective solution strategies for five challenging problems (shear-locking problem in thick plate elements, sensitivity problem to mesh distortion, non-convergence problem of non-conforming elements, accuracy loss problem of stress solutions by displacement-based elements, and singular stress problem) have also been found.

To sum up, in the contents of this book, three aspects should be emphasized:

- (1) Seven new achievements in the field of variational principle and FEM;
- (2) five new element series with 108 new element models;
- (3) five sets of novel solution strategies for five challenging problems.

The authors are very grateful to all the colleagues and students who made significant contributions to the contents included in this book. We also thank China Academy of Building Research for compiling our algorithms and finite element models into their FEM software product, SATWE, for designs of high-rise building structures.

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# Contents

**Chapter 1 Introduction—The Evolutive Finite Element Method**..... 1

- 1.1 Brief Review of the Features of Finite Element Method ..... 1
- 1.2 Finite Element Method and Variational Principles..... 3
- 1.3 Research Areas of FEM ..... 5
- 1.4 Advances in FEM and Outline of This Book ..... 6

References ..... 9

## **PART I Advances in Variational Principles**

**Chapter 2 The Sub-Region Variational Principles** ..... 15

- 2.1 Introduction ..... 15
- 2.2 The Sub-Region Variational Principle for Elasticity ..... 16
- 2.3 The Sub-Region Variational Principle for Elastic Thin Plate ..... 28
- 2.4 The Sub-Region Variational Principle for Elastic Thick Plate ..... 40
- 2.5 The Sub-Region Variational Principle for Elastic Shallow Shell ..... 51
- 2.6 The Sub-Region Mixed Energy Partial Derivative Theorem ..... 58

References ..... 64

**Chapter 3 Variational Principles with Several Adjustable Parameters**..... 66

- 3.1 Introduction ..... 66
- 3.2 Several Patterns of Functional Transformation ..... 67
- 3.3 Generalized Variational Principle Involving Several Adjustable Parameters ..... 75
- 3.4 Variable-Substitution-Multiplier Method ..... 83

References ..... 85

## **PART II Advances in Finite Element Method—Generalized Conforming Elements**

**Chapter 4 Generalized Conforming Element Theory** ..... 89

- 4.1 Introduction ..... 89
- 4.2 Conforming and Nonconforming Elements—Some Consideration about “Conforming” ..... 90
- 4.3 The First Pattern of Generalized Conforming Element—Replacing Nodal Conforming by Line Conforming Conditions..... 91

4.4	The Variational Basis of Generalized Conforming Element—Duality ...	94
4.5	The Synthesis of Energy Method and Weighted Residual Method —Flexibility .....	97
4.6	The Convergence of Generalized Conforming Element .....	99
	References .....	99

**Chapter 5 Generalized Conforming Thin Plate Element I**

—	<b>Introduction</b> .....	101
5.1	Introduction .....	101
5.2	The Generalized Conforming Conditions and Their Equivalent Forms for Thin Plate Elements .....	102
5.3	General Formulations of the Generalized Conforming Thin Plate Elements .....	105
5.4	Several Construction Schemes of the Generalized Conforming Thin Plate Elements .....	107
5.5	A Collection of the Recent Generalized Conforming Thin Plate Elements .....	111
	References .....	118

**Chapter 6 Generalized Conforming Thin Plate Element II**

—	<b>Line-Point and SemiLoof Conforming Schemes</b> .....	120
6.1	Line Conforming Scheme—Elements TGC-9 and TGC-9-1 .....	120
6.2	Line-Point Conforming Scheme—Rectangular Elements .....	130
6.3	Line-Point Conforming Scheme—Triangular Elements .....	146
6.4	Super-Basis Line-Point Conforming Scheme—Elements GCIII-R12 and GCIII-T9 .....	155
6.5	Super-Basis Point Conforming Scheme—Elements MB1-T9 and MB2-T9 .....	164
6.6	SemiLoof Conforming Scheme .....	167
	References .....	174

**Chapter 7 Generalized Conforming Thin Plate Element III**

—	<b>Perimeter-Point and Least-Square Conforming Schemes</b> .....	176
7.1	Perimeter-Point Conforming Scheme—Elements LR12-1 and LR12-2 .....	176
7.2	The Application of Perimeter Conforming Conditions—Verification for the Convergence of the Element ACM .....	181
7.3	Super-Basis Perimeter-Point Conforming Scheme—Verification and Improvement of the Element BCIZ .....	187
7.4	Least-square Scheme—Elements LSGC-R12 and LSGC-T9 .....	198
	References .....	202

<b>Chapter 8 Generalized Conforming Thick Plate Element</b> .....	203
8.1 Summary of the Thick Plate Theory .....	203
8.2 Comparison of the Theories for Thick Plates and Thin Plates .....	215
8.3 Thick/Thin Beam Element .....	232
8.4 Review of Displacement-based Thick/Thin Plate Elements .....	235
8.5 Generalized Conforming Thick/Thin Plate Elements (1) —Starting with Assuming $(\psi, \gamma)$ .....	237
8.6 Generalized Conforming Thick/Thin Plate Elements (2) —Starting with Assuming $(w, \gamma)$ .....	249
8.7 Generalized Conforming Thin/Thick Plate Elements —From Thin to Thick Plate Elements.....	260
References.....	266
<b>Chapter 9 Generalized Conforming Element for the Analysis of the Laminated Composite Plates</b> .....	268
9.1 Introduction.....	268
9.2 Fundamental Theory .....	270
9.3 New Element CTMQ20 for the Analysis of Laminated Composite Plates .....	275
9.4 The Hybrid-Enhanced Post-Processing Procedure for Element Stresses.....	286
9.5 Vibration Analysis of Laminated Composite Plates.....	290
9.6 Numerical Examples .....	292
References.....	301
<b>Chapter 10 Generalized Conforming Element for the Analysis of Piezoelectric Laminated Composite Plates</b> .....	304
10.1 Introduction.....	304
10.2 The First-Order Shear Deformation Theory of Piezoelectric Laminated Composite Plate.....	306
10.3 New Piezoelectric Laminated Composite Plate Element CTMQE ....	309
10.4 The “Partial Hybrid”-Enhanced Post-Processing Procedure for Element Stresses.....	314
10.5 Numerical Examples .....	318
References.....	323
<b>Chapter 11 Generalized Conforming Membrane and Shell Elements</b> .....	325
11.1 Introduction .....	325
11.2 Generalized Conforming Isoparametric Membrane Element.....	326
11.3 Membrane Elements with Drilling Freedoms—Definition of the Drilling Freedom and the Corresponding Rectangular and Quadrilateral Elements .....	334

11.4	Membrane Elements with Drilling Freedoms—Triangular Elements...	346
11.5	Flat-Shell Elements—Triangular Thick/Thin Shell Element GMST18.....	357
11.6	Shallow Shell Element—Variational Principle and Membrane Locking Problem .....	370
11.7	Shallow Shell Element—Triangular Element SST21 with Mid-Side Nodes.....	375
11.8	Shell Element for Geometrically Nonlinear Analysis —Triangular Flat-Shell Element GMST18 .....	382
11.9	Shell Element for Geometrically Nonlinear Analysis —Rectangular Shallow Shell Element SSR28 .....	386
	References .....	398

### **PART III Other Advances in Finite Element Method**

<b>Chapter 12</b>	<b>Sub-Region Mixed Element I—Fundamental Theory and Crack Problem .....</b>	<b>405</b>
12.1	Review of the Sub-Region Mixed Element Method .....	405
12.2	Basic Equations of the Sub-Region Mixed Element Method.....	408
12.3	2D Crack Problem.....	411
12.4	Cracked Thick Plate Problem.....	418
12.5	Surface Crack Problem in a 3D Body .....	426
	References .....	435
<b>Chapter 13</b>	<b>Sub-Region Mixed Element II—V-Notch Problem .....</b>	<b>438</b>
13.1	Introduction .....	438
13.2	Plane V-Notch Problem.....	438
13.3	Plane V-Notch Problem in a Bi-Material .....	450
13.4	Anti-Plane V-Notch Problem in a Bi-Material .....	457
13.5	V-Notch Problem in Reissner Plate.....	463
13.6	3D V-Notch Problem.....	481
	References .....	493
<b>Chapter 14</b>	<b>Analytical Trial Function Method I—Membrane and Plate Bending Elements.....</b>	<b>495</b>
14.1	Recognition of the Analytical Trial Function Method.....	495
14.2	4-Node Membrane Elements Based on the Analytical Trial Function Method .....	498
14.3	Avoiding Trapezoidal Locking Phenomenon by ATF Elements.....	500
14.4	The Basic Analytical Solutions of the Thick Plate Theory and ATF Elements Free of Shear Locking .....	504
14.5	Development of Quadrilateral Thin-Thick Plate Element Based on the Analytical Trial Function Method.....	506

14.6 Analytical Trial Function Method for Developing a Triangular Thick Plate Element Based on a Thin Plate Element .....	510
References .....	516
<b>Chapter 15 Analytical Trial Function Method II— Singular Elements</b>	
<b>with Crack and Notch .....</b>	<b>518</b>
15.1 Introduction .....	518
15.2 The Basic Analytical Solutions of the Plane Crack Problem .....	519
15.3 Element ATF-MS with Crack Formulated by the Analytical Trial Function Method .....	523
15.4 Error Analysis of Element ATF-MS with Crack .....	525
15.5 Analysis of Zero Energy Mode in Element and in Structural System .....	529
15.6 The Basic Analytical Solutions of the Plane Notch Problem .....	535
15.7 Element ATF-VN with Notch Formulated by the Analytical Trial Function Method .....	538
15.8 Error Analysis of Element ATF-VN with Notch .....	542
References .....	545
<b>Chapter 16 Quadrilateral Area Coordinate Systems, Part I</b>	
<b>—Theory and Formulae .....</b>	<b>546</b>
16.1 Introduction .....	546
16.2 The Isoparametric Coordinate Method and the Area Coordinate Method .....	547
16.3 Two Shape Characteristic Parameters of a Quadrilateral .....	549
16.4 The Definition of Quadrilateral Area Coordinates (QACM- I) .....	553
16.5 Two Identical Relations Among Area Coordinates (QACM- I) .....	556
16.6 Transformation Relations Between the Area Coordinate System (QACM- I) and the Cartesian or Isoparametric Coordinate System ..	558
16.7 Differential Formulae (QACM- I) .....	560
16.8 Integral Formulae (QACM- I) .....	562
16.9 The Proof of the Basic Formulae (A) and (B) (QACM- I) .....	565
16.10 The Proof of the Basic Formulae (C) (QACM- I) .....	569
16.11 The Quadrilateral Area Coordinate System with Only Two Components (QACM- II) .....	570
References .....	580
<b>Chapter 17 Quadrilateral Area Coordinate Systems, Part II</b>	
<b>—New Tools for Constructing Quadrilateral Elements .....</b>	<b>582</b>
17.1 Introduction .....	582
17.2 Sensitivity Analysis of Isoparametric Elements to Mesh Distortion ..	583
17.3 Brief Review of the Finite Element Models Formulated by Quadrilateral Area Coordinate Methods .....	586

17.4	4-Node Quadrilateral Membrane Elements Formulated by the Area Coordinate Method .....	589
17.5	Geometrically Nonlinear Analysis Using Element AGQ6-I .....	601
17.6	Quadrilateral Membrane Elements with Drilling Degrees of Freedom Formulated by the Area Coordinate Method .....	606
17.7	8-Node Quadrilateral Membrane Elements Formulated by the Area Coordinate Method .....	613
17.8	Quadrilateral Thin Plate Element Formulated by the Area Coordinate Method.....	620
17.9	Quadrilateral Thick Plate Element Formulated by the Area Coordinate Method.....	628
17.10	Quadrilateral Laminated Composite Plate Element Formulated by the Area Coordinate Method .....	635
	References.....	637
<b>Chapter 18 Spline Element I— Analysis of High-Rise Building Structures .....</b>		
	18.1 Introduction.....	641
	18.2 Spline Beam Elements .....	642
	18.3 Spline Plane Membrane Elements.....	646
	18.4 Analysis of Shear Wall Structures by Spline Elements.....	648
	18.5 Analysis of Frame-Tube Structures by Spline Elements.....	655
	References.....	661
<b>Chapter 19 Spline Element II— Analysis of Plate/Shell Structures .....</b>		
	19.1 Spline Elements for Thin Plate Bending .....	663
	19.2 Spline Elements for Thick/Thin Beam and Plate .....	665
	19.3 Spline Elements for Shallow Shell.....	670
	19.4 Spline Elements for Thick/Thin Shell .....	672
	19.5 Spline Elements for Geometrically Nonlinear Analysis.....	681
	References.....	689
<b>Chapter 20 Concluding Remarks.....</b>		
	20.1 Seven New Achievements in the Finite Element Method .....	691
	20.2 Five New Element Series with 108 New Element Models .....	693
	20.3 New Solution Strategies for Five Challenging Problems.....	699
	References.....	700
<b>Appendix .....</b>		
	A The equivalent equation of the functional stationary condition (2-45)....	703
	B The node conditions derived from the stationary condition (2-77) .....	704
	C $l_{ij}$ and $\gamma_{ij}$ in Eq. (13-137).....	705
	D $s_{ij}$ and $t_{ij}$ in Eq. (13-144).....	706